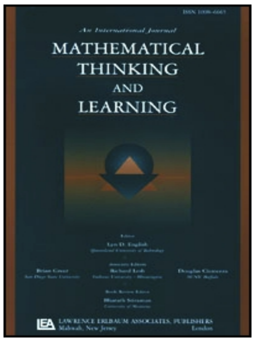

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Attributes of Instances of Student Mathematical Thinking that Are Worth Building on in Whole-Class Discussion

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ABSTRACT

This study investigated attributes of 278 instances of student mathematical thinking during whole-class interactions that were identified as having high potential, if made the object of discussion, to foster learners' understanding of important mathematical ideas. Attributes included the form of the thinking (e.g., question vs. declarative statement), whether the thinking was based on earlier work or generated in the moment, the accuracy of the thinking, and the type of thinking (e.g., sense-making). Findings illuminate the complexity of identifying student thinking worth building on during whole-class discussion and provide insight into important attributes of these high potential instances that could be used to help teachers more easily recognize them. Implications for researching, learning, and enacting the teaching practice of building on student mathematical thinking are discussed.

For several decades reform documents (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000, 2014) have consistently called for teaching that focuses on developing students' abilities to reason mathematically. For mathematical reasoning to occur, the NCTM recommends that students explore complex tasks, state and test conjectures, and build arguments to justify those conjectures. In response to these recommendations, many researchers have investigated issues around student thinking, such as students' abilities to think mathematically using tasks with high cognitive demand (Stein, Grover, & Henningsen, 1996), obstacles to students' learning (Bishop et al., 2014), challenges beginning teachers face when trying to use student thinking (Peterson & Leatham, 2009), and important teachable moments created by student thinking made public during classroom instruction (Stockero & Van Zoest, 2013). Little is known, however, about the nature of student thinking that becomes publicly available for teachers to use during instruction.

Our ongoing work investigates student mathematical thinking[1] made public during whole-class interactions that, if made the object of discussion, has the potential to foster learners' understanding of important mathematical ideas—instances of student thinking that Leatham, Peterson, Stockero, and Van Zoest called **M**athematically Significant Pedagogical **O**pportunities to Build on **St**udent **T**hinking [MOSTs] (Leatham et al., 2015). The work reported here analyzes instances of student thinking that have been identified as MOSTs in order to investigate attributes of this high-potential subset of student thinking. A better understanding of the attributes of MOSTs has the potential to support both research on mathematics teaching and teacher education by informing work related to

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Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/hmtl.

[1]We recognize the "impossibility of directly accessing the thoughts of students" (Leatham et al., 2015, p. 93) and use the phrase "student (mathematical) thinking" to refer to evidenced-based inferences about student thinking based on what students say and do.

(a) characterizing student mathematical thinking and (b) supporting instruction centered on student thinking. In the next section we discuss the relationship of our work to these two areas of research and conclude with the research question that guided the study reported in this manuscript.

Related literature

Characterizing student mathematical thinking

Early characterizations of student mathematical thinking focused on incorrect student thinking that arose via "buggy algorithms" (e.g., Brown & VanLehn, 1980 for subtraction; Resnick et al., 1989 for decimal fractions). Attention then shifted to frameworks that characterized students' thinking as they developed cognitive abilities in particular areas of mathematics (e.g., Carpenter, 1985 for addition and subtraction; Jones, Langrall, Thornton, & Mogill, 1997 for probability). This work was instrumental for providing information in these specific mathematical content areas that teachers can use to recognize patterns in what students are learning and determine appropriate directions to take instruction in response. However, it does not provide teachers with information about which student mathematical thinking in any mathematical context can be most productively used to further student understanding in the moment, particularly given the fluid and often unanticipated nature of classroom discourse.

In response to this need to provide knowledge about which student thinking is worth pursuing in the moment it surfaces, researchers have begun to consider characteristics of student mathematical thinking that is made public during instruction. For example, Stockero and Van Zoest (2013) examined student thinking during whole-class discussion to investigate instances in beginning teachers' classrooms that met their definition of a pivotal teaching moment (PTM)—"an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 127). They found that PTMs with high potential to support student learning could be categorized into just five types: (1) *extending*—students make connections to ideas beyond the lesson; (2) *incorrect mathematics*—incorrect student mathematical thinking becomes public; (3) *sense making*—students are trying to make sense of the mathematics under consideration; (4) *contradiction*—student responses have competing interpretations or conflict with one another in some way; and (5) *mathematical confusion*—students clearly state mathematically what they are confused about. They suggested that these types might be useful in helping teachers learn to recognize which interruptions in a lesson might be worth taking time to pursue. In a parallel study with experienced mathematics teachers, Sun and Hanna (2013) found that the PTM types identified by Stockero and Van Zoest were sufficient to categorize the PTMs in their data. Findings from these two studies suggest that regardless of a teacher's level of experience, identifiable types of student mathematical thinking emerge in their classrooms and create opportunities for student learning.

Research on MOSTs (e.g., Leatham et al., 2015) extends the PTM work by providing a framework for analyzing all instances of student mathematical thinking that occur during instruction—not only those that involve an interruption in the flow of the lesson—to determine which instances of student thinking have substantial mathematical and pedagogical potential. The MOST Analytical Framework (discussed in more detail in the Theoretical Framework section) provides a set of criteria for identifying a particularly important subset of student mathematical thinking that surfaces during classroom instruction—thinking that, if made the object of discussion by the class, has a high likelihood of improving the class's understanding of important mathematics. Our work characterizing attributes of MOSTs provides additional insight into these high-potential instances of student mathematical thinking.

Supporting instruction centered on student thinking

Using student mathematical thinking productively requires that the thinking be *noticed* (van Es & Sherin, 2002). Well before the construct of teacher noticing took hold in mathematics teacher education, work related to Cognitively Guided Instruction (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) focused on improving teachers' ability to recognize important student thinking by giving teachers access to different strategies students employ to solve addition and subtraction problems. They found that knowledge of student strategies positively affected teachers' tendency to carefully *attend* to the processes students used, thus contributing to their ability to adapt instruction in response to students' current thinking—the result of *interpreting* the thinking and *deciding* how to respond to it. Attending, interpreting, and deciding were defined by Jacobs, Lamb, and Philipp (2010) as component skills of noticing and are widely recognized as such.

Recent interventions have focused on helping teachers develop the skills to notice important student ideas, with numerous studies documenting success in developing such skills (e.g., Roth McDuffie et al., 2014; Sherin & van Es, 2009; Stockero, 2014). To improve manageability, many of these interventions have restricted what is available to notice in some way, such as through the use of short video clips (Schack et al., 2013; van Es, 2011). Further, these interventions often limit the development of noticing to just area of mathematics, such as early arithmetic reasoning (Schack et al., 2013) or algebraic thinking (Walkoe, 2015). The work of teaching, however, requires that teachers notice student ideas as they unfold during complex classroom interactions involving a range of mathematical topics. Thus, there is a need for tools to support teachers in noticing and using important student thinking across this full range of topics.

In the context of students engaging in discussion around a high-level task, Smith and Stein (2011) developed an approach to support teachers to orchestrate classroom discussion effectively—the five practices of *anticipating, monitoring, selecting, sequencing*, and *connecting* (p. 8) student work that has potential to enhance learning in a developmentally appropriate way. Although this approach increases the likelihood of a productive discussion about student thinking that teachers have monitored and selected in advance of whole-class discussion, there are many student ideas that teachers do not have an opportunity to consider before they are made public; for example, when students ask a question while the teacher is working an example or when students respond to other students' ideas. The challenge of noticing which instances of student thinking to pursue may be mitigated by providing teachers with a means to distinguish among the many student ideas that surface during a lesson.

The MOST Analytic Framework (Leatham et al., 2015) is a teacher education tool designed to support teachers in recognizing which in-the-moment thinking is most productive to pursue. In fact, initial efforts to use variations of this framework during interventions that use whole-class video from a range of mathematics classrooms indicate the potential of the framework to support the development of teachers' skills in noticing important student mathematical thinking (Stockero, 2014; Stockero, Rupnow, & Pascoe, 2015). Understanding attributes of MOSTs would provide information about the nature of high-potential student mathematical thinking that might be available to teachers in their classrooms. This information could be used by teacher educators to better equip teachers to notice student ideas that have significant potential to support student learning of important mathematics, both as they monitor student work and as they consider in-the-moment ideas that surface during class discussions. Thus, identifying attributes of MOSTs has the potential to support the improvement of teachers' ability to orchestrate classroom discussion that fosters student learning.

Research question

We see analysis of MOSTs as a means toward identifying important attributes of high-potential student mathematical thinking—attributes that might be used to help support teachers in developing the practice of productively using such thinking. In this study, we analyze instances that have been identified as MOSTs to investigate the following research question: *What are the attributes of MOSTs that are found in secondary mathematics classrooms?*

Theoretical framework

Leatham and colleagues (2015) defined MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking—as occurring in the intersection of three critical characteristics of classroom instances: student mathematical thinking, significant mathematics, and pedagogical opportunities. For each characteristic, two criteria were provided to determine whether an instance of student thinking embodies that characteristic. For student mathematical thinking the criteria are "(a) one can observe student action that provides sufficient evidence to make reasonable inferences about *student mathematics* and (b) one can articulate a mathematical idea that is closely related to the student mathematics of the instance—what we call a *mathematical point*" (p. 92). Student mathematics (SM) is defined as a clearly articulated statement of an inference of what a student has expressed mathematically. A mathematical point (MP) is "the [well-specified statement of a mathematical truth] that (1) students could gain from considering a particular instance of student thinking and (2) is most closely related to the SM of the thinking" (Van Zoest, Stockero, Leatham, & Peterson, 2016).

The criteria for significant mathematics are: "(a) the mathematical point is *appropriate* for the mathematical development level of the students and (b) the mathematical point is *central* to mathematical goals for their learning" (Leatham et al., 2015, p. 96). To be appropriate the MP needs to be "accessible to the students given their prior mathematical experience," (p. 96) but not likely to be already understood. Centrality could be related to the lesson, unit, course or the discipline of mathematics. Finally, "an instance embodies a pedagogical opportunity when it meets two key criteria: (a) the student thinking of the instance creates an *opening* to build on that thinking toward the mathematical point of the instance and (b) the *timing* is right to take advantage of the opening at the moment the thinking surfaces during the lesson" (p. 99). An opening is defined as "an instance in which the expression of a student's mathematical thinking seems to create, or has the potential to create, an intellectual need for students to make sense of the student mathematics" (p. 99). Timing refers to pedagogical timing, rather than the time on the clock. When an instance satisfies all six criteria, it embodies the three requisite characteristics and is a MOST.

MOSTs are instances of student thinking worth *building* on—student thinking worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea (the mathematical point of the instance) (Leatham et al., 2015). Building encapsulates core ideas of current thinking about effective teaching and learning of mathematics that include the following: "mathematics learning [is] an active process, in which each student builds his or her own mathematical knowledge from personal experiences, coupled with feedback from peers, teachers and other adults, and themselves" (NCTM, 2014, p. 9), students "construct knowledge socially, through discourse, activity, and interaction related to meaningful problems" (p. 9), and "effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (p. 10). Building on MOSTs is a particularly productive way for teachers to engage students in meaningful mathematical learning because it incorporates these ideas simultaneously. Conversely, *not* building on MOSTs risks both missing a rich learning opportunity and undermining another core idea of quality mathematics instruction—that of positioning students as legitimate mathematical thinkers (e.g., NCTM, 2014).

Methodology

In the following sections we first describe the data used in the study and the ways these data were analyzed. We then provide five illustrative MOSTs from the data set and use them to introduce the coding scheme we developed to analyze the attributes of MOSTs. We conclude this methodology section by connecting the SM, MP, and attribute coding of the five illustrative MOSTs.

Data

This study is part of a larger project focused on understanding what it means for teachers to build on students' mathematical thinking during classroom instruction (see LeveragingMOSTs.org). The larger MOST project intentionally recruited sixth through twelfth grade mathematics teachers whose classrooms reflected the diversity of teachers, students, mathematics, and curricula present in US schools. To avoid research skewed toward any particular school setting, we developed a matrix that included different aspects of teacher diversity (race/ethnicity, gender, experience, teaching style), mathematics and curricular diversity (grade, topic, textbook), and classroom diversity (region of the United States, community type, race/ethnicity), and deliberately collected videotapes of lessons to represent these aspects in the data. The resulting 11 videotaped mathematics lessons that form the data for this study range from sixth grade mathematics to Advanced Placement Calculus. They are from places as diverse as California, Hawaii, Michigan, Mississippi, New Mexico, and Utah, and from classrooms that included racial and ethnic diversity, as well as from classrooms where the teachers and students were from the same dominant or often-marginalized group. The sample was not large enough to make definitive claims about MOSTs in different types of classrooms, thus we do not provide the specifics of each lesson. The purpose of the careful attention to representativeness was to increase the generalizability of the results.

Data analysis

The unit of analysis was an *instance* of student thinking—an "observable student action or small collection of connected actions" (Leatham et al., 2015, p. 92)—that had the potential to be mathematical. Although we believe that MOSTs can occur in any classroom context (i.e., whole group, small group, paired or individual work), for this analysis we considered only instances of student mathematical thinking that occurred during whole-class interactions. Because these instances are part of the public discourse, they are readily available for the teacher to use to advance the learning of the whole class and thus seem a fruitful focus. Studiocode (Sportstec, 1997–2016) video analysis software was used for three passes of coding.

In Pass 1, classroom context was noted and instances of student thinking were identified and transcribed. Pass 1 analysis was completed by a group of undergraduate mathematics education students who individually identified instances of student thinking and met to reconcile them. They were instructed to include anything that had the potential to be student mathematical thinking and met regularly with principal investigators to discuss any issues that they encountered.

During Pass 2, the MOST Analytic Framework (see Leatham et al., 2015) was used to determine which instances of student thinking were MOSTs. After individually coding the instances for Pass 2 using the MOST Analytic Framework, the principal investigators and research assistants met to reconcile the codes. At least three principal investigators were involved in each Pass 2 reconciling session. At the conclusion of these two passes we were able to analyze the frequency of instances of student mathematical thinking and of MOSTs in each videotaped lesson.

Pass 3 focused on characterizing attributes of the MOSTs that were identified in Pass 2. We used a *grounded theory* (Strauss & Corbin, 1990) approach to develop the coding structure for analyzing MOST attributes; that is, we used *open coding* to identify potential attribute codes, and then revisited and refined the codes on subsequent passes through the data. Once we had a stable coding structure, three research assistants individually coded each MOST and then reconciled the coding as a group. If they were not able to reach agreement about the coding, the issue was brought to the attention of the principal investigators and either the codes or the code definitions were modified to resolve the issue. We then used *axial coding* (Strauss & Corbin, 1990) to relate the codes to each other. This coding process resulted in seven attributes divided into two groups: *Context attributes* and *Student Mathematics (SM) attributes*. The Context attributes locate a MOST within the mathematical and lesson terrain. The SM attributes focus on the expression of the student's mathematical thinking. These attributes are discussed in further detail in the *Context Attributes* and *Student Mathematics (SM) Attributes* sections.

Five illustrative MOSTs

We now present five MOSTs (identified during the Pass 2 coding) that we subsequently use to illustrate our data analysis procedures.[2] Because this analysis requires knowing the context in which the MOST occurred as well as the student mathematics (SM) and mathematical point (MP) of the instance, we first provide these details about each of these MOSTs.

MOST 1 occurred in a grade 7 lesson on linear relationships that was based on this problem: "For Susan's birthday she got $25 from her grandmother. Instead of spending the money she was going to save the money. She put it in the bank, and she saved an additional $2.50 every week from babysitting." After the students had been engaged in small groups in making a graph, table and equation for the situation, the teacher began a whole-class discussion of their work by asking the class if anyone had had a problem making the table or graph. A student responded, "I got confused on where to put the weeks on the graph." The teacher asked that student, "So which one did you put where?" and the student replied, "I put the money on the bottom and weeks on the side" (MOST 1). The inferred SM of MOST 1 is, "I put the money on the x-axis and weeks on the y-axis" and the related MP is, "The placement of the variables on the axes of a graph is determined by what makes the most sense in the problem situation given the established convention of the x-axis representing the independent variable."

MOST 2 occurred later in the same lesson as MOST 1, when the students were discussing the slope of the linear model of the situation. The teacher asked a student to clarify what they meant by their statement that the slope "[is] not getting faster" and the student responded, "It's going up by a constant rate. It's not going any faster at any level. Like it keeps on going up two-fifty" (MOST 2). The inferred SM of MOST 2 is, "The slope is: (1) increasing at a constant rate; (2) not going any faster; and (3) always going up $2.50" and the related MP is, "The difference between an increasing graph and an increasing slope is that on an increasing graph the values of the dependent variable increase as the values of the independent variable increase while an increasing slope occurs when the rate of change increases with each increase in the independent variable."

MOST 3 occurred later in the same lesson as MOSTs 1 and 2. The teacher asked the class who could use the graph of the equation $y = 2.5x + 25$ that was on the board to sketch what the graph would look like if Susan was able to save $5 every week from babysitting instead of $2.50. The teacher selected a student who volunteered. The student went up to the board, picked up a different color marker and said, "Okay, I'm pretty sure it still starts at 25 dollars." (MOST 3). The inferred SM of MOST 3 is, "The graph still starts at $25 when we are saving $5 a week." and the related MP is, "Changing the rate of change of a linear function without changing the initial value will leave the y-intercept unaffected."

MOST 4 occurred in a grade 10–12 lesson in which the teacher was reviewing the previous day's homework about solving proportions. As various solutions were discussed, the difference between the actions of *solving* and *simplifying* arose as an issue. A possible distinction under consideration was that solving involved an equals sign while simplifying did not. A student raised her hand to get the teacher's attention and said, "Doesn't solving sometimes include simplifying?" (MOST 4). The inferred SM of MOST 4 is, "Doesn't solving sometimes include simplifying?" and the related MP is, "Simplification is a process of dividing out common factors in the numerator and denominator of fractions and combining like terms that, when used on equivalent expressions (equations), can lead to solving for the variable."

MOST 5 occurred in a high school geometry lesson on deriving the formulas for the surface area and volume of a sphere. As the students were discussing their findings, one of them asked what the units would be for surface area. After the teacher asked about the dimension of surface area and a student said, "Two," the teacher asked the class, "So what would the units be?" Various students

[2]Note that these MOSTs were selected to illustrate our data analysis procedures, not our data set. Thus, to reduce the number of contexts that needed to be described, some of the MOSTs are from the same lesson.

SM & MP for Five MOSTs			Coding	
#	**Student Mathematics (SM)**	**Mathematical Point (MP)**	**Context Attributes (*Prompt*, *Basis*, *Math Goal*)**	**SM Attributes (*Form*, *Accuracy*, *Transparency*, *Type*)**
1	I put the money on the x-axis and weeks on the y-axis.	The placement of the variables on the axes of a graph is determined by what makes the most sense in the problem situation given the established convention of the x-axis representing the independent variable.	*Targeted Invitation* *Pre-Thought* *Course*	*Tentative Statement* *Incorrect* *Obvious* *Incorrect or Incomplete*
2	The slope is: (1) increasing at a constant rate; (2) not going any faster; and (3) always going up $2.50.	The difference between an increasing graph and an increasing slope is that on an increasing graph the values of the dependent variable increase as the values of the independent variable increase while an increasing slope occurs when the rate of change increases with each increase in the independent variable.	*Targeted Invitation* *In-the-Moment* *Discipline*	*Declarative Statement* *Combination* *Hidden* *Sense Making*
3	The graph still starts at $25 when we are saving $5 a week.	Changing the rate of change of a linear function without changing the initial value will leave the y-intercept unaffected.	*Open Invitation Selected* *In-the-Moment* *Lesson*	*Tentative* *Correct* *Translucent* *Information*
4	Doesn't solving sometimes include simplifying?	Simplification is a process of dividing out common factors in the numerators and denominators of fractions and combining like terms that, when used on equivalent expressions (equations), can lead to solving for the variable.	*Spontaneous* *In-the-Moment* *Course*	*Question* *N/A* *Translucent* *Sense Making*
5	The unit of measure for surface area is either squared or cubed.	The unit of measure for a two-dimensional measurement is square units.	*Open Invitation Selected* *In-the-Moment* *Unit*	*Declarative Statement* *Combination* *Obvious* *Multiple Ideas or Solutions*

Figure 1. The SM, MP and attribute coding of five illustrative MOSTs.

responded simultaneously with “cubed” and “squared” (MOST 5). The inferred SM of MOST 5 is, “The unit of measure for surface area is either squared or cubed.” and the related MP is, “The unit of measure for a two-dimensional measurement is square units.”

Figure 1 provides a summary of the five illustrative MOSTs, highlighting the student mathematics and the associated mathematical point of each MOST. It also includes the Context and Student Mathematics (SM) attribute coding that is discussed in the following sections.

Context attributes

The Context attributes locate a MOST within the mathematical and lesson terrain. These attributes describe the invitation, or lack thereof, that precipitated the MOST (Prompt), whether the student mathematics in the MOST was based on earlier work or in-the-moment thinking (Basis), and the distance of the mathematical idea of the MOST from the day’s lesson goals (Mathematical Goal). Figure 2 provides a summary of the Context attributes and their definitions, which are elaborated next. Each MOST was coded for each of the three Context attributes.

Prompt refers to the invitation (Targeted Invitation, Open Invitation Selected, and Open Invitation Spontaneous) or lack thereof (Spontaneous) that precipitated the MOST. MOSTs 1 and 2 (see *Five Illustrative MOSTs* or Figure 1) were prompted by the teacher’s requests to particular students to share their thinking and thus were coded as Targeted Invitation. In contrast, MOST 4 had no obvious prompt and thus was coded Spontaneous. MOSTs 3 and 5 arose when the teacher

Attribute	Code	Definition
Prompt	Targeted Invitation	The utterance is prompted by a teacher's (or student's) request to a specific student or group to share their thinking or answer a question.
	Open Invitation Selected	The prompt is an open invitation and a student or group is called on to share their answer.
	Open Invitation Spontaneous	The prompt is an open invitation and a student(s) shares their thinking without being called on.
	Spontaneous	There is no obvious invitation.
Basis	Pre-Thought	The instance stems from a student(s) sharing their thinking from previous work.
	In-the-Moment	The instance stems from a student's in-the-moment thinking.
Math Goal	Lesson	The MP is related to a mathematical goal of the lesson being taught.
	Unit	The MP is not the focus of the lesson being taught, but is closely enough related to the content of the lesson that one would expect it to be a goal for the unit that includes the lesson.
	Course	The MP is likely to be a goal at some point in the course in which the lesson takes place (e.g., Algebra, Calculus, Integrated Mathematics 1, etc.).
	Discipline	The MP is a broader goal for the students as learners of mathematics.

Figure 2. Coding guide for context attributes.

invited the class to participate and then called on a student to share their thinking, so they received the code Open Invitation Selected.

Basis refers to whether the SM of the MOST is based on earlier work (Pre-Thought) or on in-the-moment thinking (In-the-Moment). All five MOSTs came from class discussions about tasks students had solved beforehand in small groups. MOST 1 was a reporting out of a student's earlier work and thus received the code Pre-Thought. The remaining four MOSTs were in response to what was currently being shared, rather than something students had done earlier, thus they were coded In-the-Moment.

Mathematical Goal refers to how close the mathematical idea captured in the MOST is to the day's lesson. A MOST's articulated MP is used to determine whether the MOST is closely linked to a mathematical goal for (a) the Lesson, (b) the Unit, (c) the Course, or (d) the discipline of mathematics (Discipline). As discussed in Leatham and colleagues (2015), "[m]athematical goals for student learning could be determined by the teacher or by an external source, such as curriculum documents (e.g., NCTM, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), or they could be inferred by an observer who is knowledgeable in the field of mathematics education" (p. 97). It was through this last means of determination that the MP analysis for this work took place. The MP in MOST 3 was at the heart of the lesson on linear relationships, thus was coded Lesson. The MP in MOST 5—two dimensional objects have square units—was not the focus of the lesson, but was closely related to the lesson's focus and part of an earlier lesson in the unit, so this MOST was coded Unit. Both MOST 1 and MOST 4 have MPs related to ideas in the course, though not in the lesson or even the unit, thus they were coded Course. The mathematical idea captured in MOST 2—the difference between an increasing graph and an increasing slope—was not an explicit goal of the lesson, unit or course, but is an important distinction for learners of these mathematical ideas to make, thus this MOST was coded Discipline.

Student Mathematics attributes

The SM attributes focus on the expression of the student's mathematical thinking. These attributes describe whether the MOST was a question or statement (Form), the validity of the student mathematics (Accuracy), the extent to which the intellectual need generated by the MOST is transparent (Transparency), and the nature of the student thinking that made the instance a MOST (Type). Figure 3 provides a summary of the SM attributes and their definitions, which are elaborated on below. Each MOST was coded for each of the four SM attributes.

Form refers to the way the student thinking is expressed (Question, Tentative Statement or Declarative Statement), regardless of its correctness or completeness. MOST 4 was a question, thus was coded Question. MOST 1 was coded Tentative Statement because the student expressed uncertainty. Other MOSTs that would be coded Tentative Statement include those in which the student adds hedges such as "maybe" or "I might be wrong" or, in the case of MOST 3, "I'm pretty sure." MOSTs 2 and 5 were statements without qualification, so these MOSTs received the code Declarative Statement.

Accuracy is used to categorize a MOST based on the validity of its SM. There are five categories of Accuracy: Correct, Incorrect, Incomplete, Combinations, and N/A. MOST 1 was coded as Incorrect because the class had already agreed to the convention of putting the independent variable on the x-axis and the dependent variable on the y-axis and the problem the students were exploring asked

Attribute	Code	Definition
Form	Question	The student thinking is shared as a question or with the intent to question.
	Tentative Statement	The student appears to be making a conjecture or is wondering about something.
	Declarative Statement	The students appear to be confident in what they are saying.
Accuracy	Correct	The SM is a correct mathematical statement.
	Incorrect	The SM is an incorrect statement.
	Incomplete	The SM is not incorrect, but has gaps or ambiguities that keep it from being completely correct.
	Combination	The SM is made up of distinct ideas that receive different *Correct, Incorrect* or *Incomplete* codes.
	N/A	The correctness of the SM cannot be determined (e.g., it is a question).
Transparency	Obvious	The SM itself highlights the problematic aspect of the mathematics; no work needs to be done by the teacher to reveal the problematic aspect of the mathematics.
	Translucent	The intellectual need presented by the SM is likely clear to only one or a few students, so some work needs to be done to make the problematic aspect of the SM apparent to others in the class, or students might recognize that there is something to engage with but would likely be unable to articulate what it is.
	Hidden	What makes the SM problematic is not likely to be visible to the students unless the teacher reveals it.
Type	Incorrect or Incomplete	It is the incorrect or incomplete nature of the SM that creates the intellectual need.
	Sense Making	The SM implies that the student was trying to make sense of the mathematics, or they had comprehended an idea with which the class had been struggling.
	Multiple Ideas or Solutions	The SM creates an opportunity for comparison of multiple ideas or solutions.
	Information	A student provides factual information critical to what the class is in the process of establishing.

Figure 3. Coding guide for *Student Mathematics (SM)* attributes.

them to find the amount of money given the number of weeks worked, implying "money" was the dependent variable and "weeks" the independent variable. MOSTs that are coded Correct are clearly correct, such as MOST 3, while statements coded incomplete do not include falsehoods, but are missing something that is needed to be clearly correct. MOSTs 2 and 5 received the code Combination because they include distinct correct and incorrect ideas. MOST 2 included the correct idea that the rate of change in this situation is a constant \$2.50, but the language suggests that the slope is *increasing* at that rate rather than that the slope *is* that constant rate. MOST 5 was coded Combination because it involved some students shouting out a true statement—the units of a two-dimensional measure are squared—and others a false statement—the units of a two-dimensional measure are cubed. MOST 4 was coded *N/A* because questions, by their very nature, do not have a truth-value.

Transparency is used to categorize the visibility of the intellectual need generated by the MOST. There are three mutually exclusive categories of Transparency: Obvious, Translucent, and Hidden. MOSTs 1 and 5 were coded Obvious because, given the context, the students were likely to note the problematic aspect of the MOST. MOSTs 3 and 4 were coded Translucent because the intellectual need inherent in the students' statements was likely clear to only a small part of the class and would require some highlighting. MOST 2 was coded Hidden because the problematic aspect of the student's statement involved distinctions in language about rates of change that the class likely had not yet encountered.

Type is used to categorize what about the SM created the intellectual need that contributed to the instance being a MOST. As a result of the grounded theory analysis of the data, we identified four Type categories: Incorrect or Incomplete, Sense Making, Multiple Ideas or Solutions, and Information. The compelling aspect of MOST 1 was that the student had expressed an incorrect idea (given the established convention of the class), thus this MOST was coded Incorrect or Incomplete. MOSTs 2 and 4 involved students grappling with a mathematical idea—the meaning of slope in MOST 2 and the difference between solving and simplifying in MOST 4—thus they were both coded Sense Making. MOST 5 was compelling because students presented contradicting ideas about the unit of a 2-dimensional measure, so this MOST was coded Multiple Ideas or Solutions. MOST 3 involved a student claiming that the starting point stayed the same when the rate at which money was being saved changed. The factual nature of the claim, without any evidence of the student engaging in sense making to arrive at it, led to a code of Information. The instance is compelling because the student's claim is spot on, but occurs at a time when its validity has not yet been established.

Results and discussion

We first present the frequencies and rates of the MOSTs in the data to provide a backdrop for the discussion of the MOST Attributes. We then consider the Context and SM attributes independently. The section concludes with a look at various interactions between attribute codes.

MOST frequencies and rates

A total of 278 MOSTs were identified in the whole-class interactions of the 11 coded videos (see Table 1). MOSTs were identified in every lesson video. On average, there were 4.5 instances of student mathematical thinking (SMT) per minute of whole-class interaction, with MOSTs occurring at an average rate of 0.8 per minute. Perhaps more interesting is the rate at which MOSTs occurred in different lessons and its relationship to the rate of occurrence of instances of SMT. The highest occurrence of MOSTs, in Lesson A, was at a rate of 1.5 MOSTs per minute—almost twice the overall average. This was also the lesson in which there was the highest number of instances of SMT, 280, and the highest rate of instances of SMT per minute, 5.8. On the other end of the spectrum, the lowest occurrence of MOSTs was at a rate of 0.1 per minute in Lesson K, where there were 38

Table 1. Student mathematical thinking instances and MOST frequencies and rates.

Lesson	Minutes in Whole- Class Interaction	Number of Student Math Thinking (SMT) Instances	SMT Instances/ Minute	Number of MOSTs	MOSTs/ Minute
A	48	280	5.8	73	1.5
B	41	176	4.3	38	0.9
C	44	217	4.9	36	0.8
D	35	198	5.7	29	0.8
E	27	122	4.5	22	0.8
F	40	165	4.1	29	0.7
G	45	205	4.6	29	0.6
H	11	30	2.7	5	0.5
I	18	110	6.1	8	0.4
J	25	30	1.2	8	0.3
K	15	38	2.5	1	0.1
Total	**349**	**1571**	**4.5**	**278**	0.8

instances of SMT that occurred at a rate of 2.5 instances of SMT per minute. This rate of occurrence of SMT was the second lowest in any lesson video, higher only than Lesson J, which had a rate of 1.2 instances of SMT per minute. Although these numbers might not seem vastly different at first glance, Lesson A has 15 times more MOSTs per minute than Lesson K—a finding that seems to suggest very different types of classroom interactions. Perhaps not surprisingly, the data suggest a relationship between the rate at which instances of SMT occur and the rate at which MOSTs occur, with MOSTs occurring more frequently when students have more opportunity to share their thinking. A linear regression analysis revealed the relationship *Rate of MOSTs = 0.15 (Rate of SMT) + 0.05*, with a correlation coefficient of 0.61, a fairly strong correlation.

Context attributes of MOSTs

Figure 4 provides the percentages of MOSTs in each of the categories of the three Context attributes: *Prompt, Basis*, and *Mathematical Goal*. With respect to the Prompt, 34% of the MOSTs occurred when students were deliberately called on to respond to the issues being discussed (Targeted Invitation), 9% occurred when there was an open invitation and a student or group was called upon to respond (Open Invitation Selected), and 40% of the MOSTs arose when there was an open invitation and students responded spontaneously without being called upon (Open Invitation Spontaneous). Thus 83% of the MOSTs occurred when students were invited by the teacher (or occasionally another student) to respond (open or targeted). The fact that only 17% of MOSTs occurred without an intentional prompt suggests that, at least in these classrooms, teachers played an important role in eliciting student ideas that might be fruitful for students to discuss and of which they could take ownership.

Context Attributes					
Prompt		**Basis**		**Math Goal**	
Targeted Invitation	34	Pre-Thought	19	Lesson	64
Open Invitation Selected	9	In-the-Moment	81	Unit	9
Open Invitation Spontaneous	40			Course	16
Spontaneous	17			Discipline	11

Figure 4. Percentage of MOSTs in each context attribute subcategory.

Of the 278 MOSTs in our data, 19% were reports of work that students had completed earlier in the lesson (Pre-Thought), thus were available for the teachers to identify by monitoring students as they worked. This percentage speaks to the benefit of supporting teachers in developing skills such as the *five practices for orchestrating classroom discussion* identified by Smith and colleagues (e.g., Smith & Stein, 2011). The finding that 81% of the MOSTs were based on student thinking that occurred in-the-moment during whole-class interaction suggests, however, that if teachers only focus on pre-thought student thinking they will miss the vast majority of MOSTs. This finding speaks to the importance of supporting teachers in developing skills for noticing and responding to evolving thinking.

Almost two thirds (64%) of the mathematical points of MOSTs were aligned with the content being taught currently in class (Lesson). Thus even though a clear majority of the mathematical ideas students verbalize in class might be closely related to the lesson, 36% of the MOSTs involved ideas beyond the lesson—9% were related to the Unit, 16% to the Course, and 11% to broader goals for the students as learners of mathematics (Discipline). This result suggests that to prepare teachers to take full advantage of MOSTs, teacher educators need to support teachers to focus not only on their immediate goals, but also on how the mathematical ideas in student thinking are connected to broader goals for student learning. Broadening teachers' foci would support them in recognizing opportunities to use student thinking to establish connections among ideas within lessons, units, courses and the discipline of mathematics. Such practice could contribute in significant ways to helping students to see mathematics "as a unified whole" (National Research Council, 2001, p. 293) rather than as a collection of disjoint ideas and courses.

Student mathematics attributes of MOSTs

Figure 5 provides the percentages of MOSTs in each of the four categories of SM attributes: *Form, Accuracy, Transparency*, and *Type*. In the Form category, the vast majority of the MOSTs were Declarative Statements (77%) as opposed to Questions (16%) or Tentative Statements (7%). Thus focusing only on expressions of student mathematical thinking that are intuitively suggestive of thinking worth pursing (e.g., questioning or wondering) would result in missing the majority of the opportunities to build on MOSTs.

In terms of Accuracy, the largest percentage of MOSTs were Correct (40%), with Incorrect accounting for another 24% of MOSTs and Not Applicable (e.g., questions) accounting for 19%. This is a particularly interesting finding given that the project team had initially hypothesized that it would be difficult for correct student mathematical thinking to meet the MOST criteria. Although

Student Mathematics (SM) Attributes							
Form		**Accuracy**		**Transparency**		**Type**	
Question	16	Correct	40	Obvious	55	Incorrect or Incomplete	32
Tentative	7	Incorrect	24	Translucent	35	Sense Making	50
Declarative	77	Incomplete	8	Hidden	10	Multiple Ideas or Solutions	15
		Combinations	9			Information	3
		Not Applicable	19				

Figure 5. Percentage of MOSTs in each SM attribute subcategory.

attending to incorrect ideas is likely to result in the identification of MOSTs, many MOSTs would be missed were incorrectness to be used as the sole indicator.

With respect to Transparency, the intellectual need generated by about half of MOSTs (55%) would likely be Obvious to students. For 35% and 10% of MOSTs, however, the intellectual need inherent in the student mathematics is Translucent or Hidden, respectively. Thus for nearly half of MOSTs (45%), the teacher likely would need to highlight the intellectual need for the class to productively make the MOST an object of discussion. This result points again to the active role of the teacher in orchestrating productive discourse that builds on student thinking.

Half of the MOSTs (50%) occurred when students showed evidence of grappling with a mathematical idea (Sense Making). The next highest Type category involved instances of student thinking that were Incorrect or Incomplete (32%). Student thinking that led to Multiple Ideas or Solutions being available for students to consider occurred in 15% of the MOSTs, and 3% were Information. Although Incorrect or Incomplete and Multiple Ideas or Solutions had lower frequencies than Sense Making, we hypothesize that they may be easier for teachers to recognize. Thus, it seems important for teacher educators to sensitize teachers to MOSTs that may not be easily recognizable and, in doing so, to develop the teachers' ability to attend to all three main MOST Types. The fact that all of the MOSTs were captured by just four categories and that 97% of the MOSTs were captured by three categories is encouraging as it suggests some parameters for developing teachers' abilities to notice and take advantage of MOSTs.

It is worth noting how the Attribute Type categories relate to the five pivotal teaching moment (PTM) types (Stockero & Van Zoest, 2013) mentioned earlier. The broader coding scheme of the Attribute analysis reduced the need for individual codes to distinguish among instances. For example, the PTM *extending* type was captured by the Attribute codes of Type: Sense Making and Math Goal: Unit, Course or Discipline, thus *extending* was subsumed under Sense Making. The PTM *mathematical confusion* type was also subsumed under Sense Making; if students can "articulate mathematically what they are confused about" (Stockero & Van Zoest, 2013, p. 136), there is evidence that they are trying to make sense of the mathematics. The PTM *contradiction* type was seen as a subset of Multiple Ideas or Solutions and *incorrect mathematics* was broadened to Incorrect or Incomplete. Thus the five PTM codes are roughly equivalent to the three main Attribute Types. The fourth Attribute Type, Information, involves "factual information critical to what the class is in the process of establishing" (Figure 3) thus applies only to MOSTs that are not PTMs—which, by definition, are "interruption[s] in the flow of the lesson" (Stockero & Van Zoest, 2013, p. 127).

Interactions among MOST attributes

We now discuss selected interactions between the attributes that seem to provide useful insight into the attributes of the MOSTs that teachers have available to build on in their secondary school mathematics instruction. (A complete table of all the interactions between the attributes is included as an appendix.) We first discuss the interactions of the Context attributes with the SM attributes and then discuss interactions within the SM attributes.

Interactions of context attributes with SM attributes

The interaction between *Mathematical Goal* and *Transparency* (see Table 2) is of interest because it illuminates how the distance of the mathematical point of a MOST from the day's lesson goals might affect the work a teacher needs to do to engage students with the MOST. The Transparency was coded Obvious 59% of the time when the mathematical point of the instance was related to the lesson goal and 83% of the time when the mathematical point was related to goals of the unit in which the lesson took place (for an example of a Unit, Obvious MOST, see MOST 5 in Figure 1). This means that, in general, the teacher would have to deliberately make the intellectual need visible to students well less than half the time when the mathematical point was related to the lesson or unit. When the mathematical point of the MOST was related to goals of the Course or Discipline,

Table 2. Interaction between mathematical goal and transparency.

Math Goal	Transparency		
	Obvious	Translucent	Hidden
Lesson	59	35	6
Unit	83	17	0
Course	44	36	20
Discipline	27	46	27

however, the Transparency of the intellectual need was Obvious only 44% and 27% of the time, respectively (for an example of a Course, Obvious MOST, see MOST 1 in Figure 1). For instances related to Course, the Transparency was Translucent or Hidden 56% of the time and for instances related to Discipline the Transparency was Translucent or Hidden 73% of the time. Thus, for instances related to broader mathematical goals, well more than half of the time, the teacher would need to intentionally provoke student engagement with the mathematical content of the MOST by highlighting the problematic aspect of the student mathematics.

The data show that when the mathematical point of a MOST was related to a *Mathematical Goal* of the Lesson, the *Form* of student mathematical thinking was most often a Declarative Statement (80%), with Tentative Statements and Questions making up 8% and 12% of instances, respectively (Table 3; for an example of a Tentative, Lesson MOST, see MOST 3 in Figure 1). In contrast, when the mathematical point of a MOST was related to the Mathematical Goal of the Unit, the Form was almost evenly split between Questions (46%) and Declarative Statements (50%) (for an example of a Declarative, Unit MOST, see MOST 5 in Figure 1). MOSTs related to Course and Discipline goals were more similar to those related to the lesson, in that they were most often Declarative in form (73% and 83%, respectively). These results suggest that declarative statements should not be discounted when searching for MOSTs, particularly when looking for MOSTs related to the lesson goals.

Prompt and *Type* also have interesting interactions (see Table 4). MOSTs that occurred spontaneously were Sense-Making 80% of the time (see, for example, MOST 4 in Figure 1). This is a notable difference from instances that occurred as the result of an invitation, which were more evenly split between Incorrect or Incomplete and Sense Making (Open Invitation Selected: 33% Incorrect and Incomplete and 44% Sense Making; Open Invitation Spontaneous: 39% Incorrect and Incomplete and 36% Sense Making; and Targeted Invitation: 33% Incorrect

Table 3. Interaction between mathematical goal and form.

Math Goal	Form		
	Question	Tentative	Declarative
Lesson	12	8	80
Unit	46	4	50
Course	23	4	73
Discipline	7	10	83

Table 4. Interactions of prompt with type and form.

Prompt	Type				Form		
	Incorrect or Incomplete	Sense Making	Multiple Ideas or Solutions	Information	Question	Tentative	Declarative
Spontaneous	11	80	7	2	59	0	41
Open Invitation Selected	33	44	19	4	11	22	67
Open Invitation Spontaneous	39	36	21	4	4	7	92
Targeted Invitation	33	54	12	1	12	10	78

and Incomplete and 54% Sense Making). Spontaneous MOSTs were also distributed differently than invited MOSTs in terms of the Form of the MOST. For invited MOSTs, the Form was Declarative the majority of the time (67%, 92%, and 78% for Open Invitation Selected, Open Invitation Spontaneous, and Targeted Invitation, respectively). Spontaneous MOSTs, however, were less often Declarative (41%) and more often Questions (59%). The finding that spontaneous MOSTs often arise from students' questions about the mathematics under consideration is particularly important given the emphasis on students making sense of mathematics in descriptions of effective mathematics classrooms (e.g., NCTM, 2000, 2014). Supporting teachers to recognize the relationship between spontaneous MOSTs and students' sense making—that such MOSTs can be both evidence of student sense-making and an opportunity to engage the class in making sense of important mathematics—will better position teachers to develop effective mathematics classrooms.

Finally, we consider interactions of *Basis* with the SM attributes *Form* and *Accuracy* (Table 5). Although perhaps not surprising, it is interesting to note that Pre-Thought MOSTs were Declarative in *Form* 94% of the time. Pre-thought MOSTs, however, accounted for only 19% of the MOSTs; the other 81% of the MOSTs in our data resulted from In-the-Moment thinking. These In-the-Moment instances were also mainly Declarative in *Form* (72%), with Questions accounting for another 20%. Although Pre-Thought student mathematics was more likely to be Correct than student mathematics that was generated In-the-Moment (52% to 37%), both types of student mathematics were also often Incorrect (20% and 25%, respectively). Together these findings suggest that teachers need to be supported to skillfully respond to evolving thinking, regardless of its correctness and whether it is expressed as a question or a statement.

Interactions within SM attributes

Analysis of the bidirectional interactions between *Form* and *Type* (Tables 6 and 7) revealed several interesting relationships. Almost all (98%) of the MOSTs in the form of Questions were compelling because the student was grappling to make sense of a mathematical idea (Table 6; see, for example, MOST 4 in Figure 1). The bulk of Declarative Statements were split between Incorrect or Incomplete (36%) and Sense Making (41%), and the majority of Tentative Statements were also split between

Table 5. Interactions of basis with form and accuracy.

	Form			Accuracy				
Basis	Question	Tentative	Declarative	Correct	Incorrect	Incomplete	Combination	N/A
In-the-Moment	20	8	72	37	25	9	7	22
Pre-Thought	2	4	94	52	20	6	18	4

Table 6. Interaction between form and type.

	Type			
Form	Incorrect or Incomplete	Sense Making	Multiple Ideas or Solutions	Information
Question	2	98	0	0
Tentative Statement	55	40	0	5
Declarative Statement	36	41	20	3

Table 7. Interaction between type and form.

	Form		
Type	Question	Tentative Statement	Declarative Statement
Incorrect or Incomplete	1	12	87
Sense Making	31	6	63
Multiple Ideas or Solutions	0	0	100
Information	0	14	86

these same type categories (55% Incorrect or Incomplete; 40% Sense Making). The fact that almost all questions in this data that qualified as MOSTs involved sense making suggests that questions that involve sense making are worthy of particular attention.

In addition to Declarative Statements being the most prominent form regardless of Type (Table 7), 100% of MOSTs that were compelling because they provided an opportunity for students to consider Multiple Ideas or Solutions were Declarative Statements (see, for example, MOST 5 in Figure 1). This suggests that the Multiple Ideas or Solutions MOST type tends to occur when students are confidently sharing their differing solutions. It is also interesting to note that although almost all of the Questions were Sense Making (98%, Table 6), only about one-third of Sense Making MOSTs were Questions (Table 7). This means that although questions that are MOSTs typically indicate sense making, Declarative Statements also need to be considered to identify all Sense Making MOSTs.

The interaction between *Type* and *Transparency* (see Table 8) illuminates the extent to which a teacher may need to do work to engage students with different types of MOSTs. When the student mathematics of a MOST was inaccurate or missing critical components (Incorrect or Incomplete), the Transparency was Obvious only 26% of the time (Table 8). When Multiple Ideas or Solutions were on the table or when students were engaged in Sense Making, however, the need to engage with the student mathematics would have likely been clear to other students 76% and 67% of the time, respectively, meaning that the teacher would need to make the intellectual need apparent far less often. Together, these findings suggest that capitalizing on MOSTs that involve Incorrect or Incomplete thinking is likely to require more work on the part of the teacher than capitalizing on other types of MOSTs.

Finally, the interaction between *Form* and *Accuracy* shows that of the tentatively-stated MOSTs, 45% were Incorrect and 30% were Correct (Table 9). Nearly the opposite was true of the MOSTs that were declarative statements (28% Incorrect vs. 49% Correct). This latter finding is of interest because a declarative correct statement might prompt a teacher to just move on with the lesson; instead this finding suggests that teachers need to be supported to carefully consider whether it would be worth taking the time to engage the class in a discussion about such ideas. Overall, the findings about the interactions between Form and Accuracy suggests that although there was some correlation between students' confidence in their thinking and the accuracy of it, the relationship was not strong enough to be counted on. That is, in the context of MOSTs, relying on tentative thinking to be incorrect and confident thinking to be correct would cause one to be wrong much of the time.

Table 8. Interaction between type and transparency.

	Transparency		
Type	Obvious	Translucent	Hidden
Incorrect or Incomplete	26	56	18
Sense Making	67	29	4
Multiple Ideas or Solutions	76	14	10
Information	57	14	29

Table 9. Interaction between form and accuracy.

	Accuracy				
Form	Correct	Incorrect	Incomplete	Combination	N/A
Questions	0	0	0	0	100
Tentative Statement	30	45	5	20	0
Declarative Statement	49	28	10	10	3

Implications

We began this article by claiming that a better understanding of the attributes of MOSTs has the potential to support both research on mathematics teaching and teacher education by informing work related to (a) characterizing student mathematical thinking and (b) supporting instruction centered on student thinking. We discuss the implications of this study by returning to these two areas and considering directions for future research.

Characterizing student mathematical thinking

The MOST Analytic Framework (Leatham et al., 2015) provided a means to identify MOSTs—student thinking that has the potential to be built on to foster understanding of mathematical ideas. Identifying attributes of MOSTs extended this work and contributes to ongoing efforts to develop a theory of productive use of student mathematical thinking by characterizing this particularly valuable subset of student mathematical thinking. The small number of type categories that were needed to describe the MOSTs (almost all were Incorrect or Incomplete, Sense-Making, or Multiple Ideas or Solutions) is consistent with Stockero and Van Zoest's (2013) finding that PTMs—a subset of MOSTs—were of a small number of type categories. The attribute study as a whole extends that work in two important ways, by providing a more detailed categorization of the full range of MOSTs and by categorizing MOSTs that occur in the classrooms of teachers with a range of experience levels.

Supporting instruction centered on student thinking

A major goal of this study was to reveal attributes of MOSTs that might help teacher educators to support teachers to more easily identify student thinking that has significant potential to be built upon to support student learning of important mathematics. In some respects, the work did just the opposite, highlighting the complexity of the student thinking that teachers have available during their instruction.

The results point to the need for teachers to adjust their instruction to nuances in student thinking to take full advantage of that thinking in their instruction. For example, rather than automatically moving on when a student makes a correct declarative statement, teachers need to evaluate such utterances to determine whether the class could benefit from making the student mathematics an object of discussion. Teachers need to evaluate student questions in a similar way. Based on the findings of this study, the best teacher response to a question that does not involve sense making may be a direct answer since it is unlikely that the question is a MOST. If the question does involve sense making, however, it may be a MOST, in which case a more appropriate teacher response would be to provide an opportunity for the class to join the student who asked the question in making sense of the idea. Similarly, the fact that so many MOSTs occurred in both correct and incorrect thinking also highlights the complexity of the noticing that teachers need to engage in to skillfully respond to student thinking. That is, the teacher cannot rely on the correctness of the thinking to decide whether it would be productive to pursue. In general, there is a need to support teachers to look beyond surface features of student thinking to determine the nature of their response.

The fact that 36% of the MOSTs were related to goals beyond the lesson in which they occurred suggests that teachers need to be supported to continuously consider how the mathematical ideas in student thinking are connected to broader goals for student learning. The substantial percentage of MOSTs whose intellectual need is not likely to be obvious to the students (45%) points out the importance of teachers being prepared to help the class to recognize problematic aspects of students' mathematical thinking. Together these findings point to the need for secondary teachers to have strong mathematical knowledge for teaching [MKT], especially *horizon knowledge* (Ball, Thames, &

Phelps, 2008, p. 403) and to continue throughout their careers to deepen their mathematical understanding for secondary teaching [MUST] (Heid, Wilson, & Blume, 2015). Without this knowledge and understanding, it is unlikely that teachers would be able to identify important mathematical ideas in students' mathematical thinking in order to make these ideas available for other students in the class to consider. Further, teachers would be unlikely to be able to identify opportunities to help students to make connections between expressed student mathematical thinking and broader mathematical ideas. Making those connections, however, could provide valuable opportunities for students to come to understand mathematics as a unified and connected subject.

The finding that the majority of the MOSTs in the data were based on student thinking that occurred in the moment during whole-class interaction, suggests that focusing on *monitoring* (Smith & Stein, 2011) students during work time, while important, may not adequately prepare the teacher to respond to instances that have high potential to foster understanding during whole-class interaction. Although some of these in-the-moment instances may be anticipated during the planning phase of the lesson, others may not. Thus, this finding highlights the need for teachers to develop skills for noticing and responding to student thinking that might not be expected and that they would not have an opportunity to think about in advance—learning to exploit the unexpected (Foster, 2014) rather than to kill, fear, redirect or dismiss it (Beghetto, 2013).

Consistent with Wood, Williams, and McNeal (2006), a clear implication of these findings for teachers' practice is the importance of giving students the opportunity to share their mathematical thinking publicly. MOSTs are high-potential opportunities to foster learners' understanding of important mathematical ideas. Beyond the obvious—that MOSTs can only occur when students' thinking becomes public—the relatively high linear correlation between instances of student mathematical thinking and MOSTs suggests the importance of teachers intentionally creating opportunities for students to share their thinking with the class. The fact that the vast majority of MOSTs occurred as a result of prompting points to the need for teachers to elicit student thinking as part of their instruction.

Directions for future research

There are three particularly productive areas for future research suggested by this study: comparing MOSTs with instances of student thinking that were not MOSTs, investigating ways the attribute coding framework might be used to assess teachers' instruction and influences on it, and studying professional development approaches to improving teachers' ability to productively use MOSTs.

The study reported here focused exclusively on MOSTs. Although this focus allowed us to explore in detail attributes of this particularly high-potential subset of student thinking, we were not able to answer the question of whether attributes of MOSTs are distinct from attributes of instances of student thinking in general. For example, although we know that 32% of MOSTs reflected incorrect or incomplete thinking, we do not know what percentage of non-MOSTs reflect that type of thinking. We do know that not all instances of incorrect or incomplete thinking are MOSTs—simple calculation errors, for example, are not. Knowing how often such instances are non-MOSTs might provide further insight into identifying MOSTs. Knowing the attributes of non-MOSTs might also help to interpret what we have learned about MOSTs. For example, one finding about MOSTs was that relatively few of them occurred spontaneously (without any invitation from the teacher). Also knowing the percentage of non-MOSTs that occurred spontaneously would allow us to conclude whether the low number of spontaneous MOSTs pointed to the important role teachers play in eliciting MOSTs or simply reflected the number of times students spontaneously expressed their thinking in general. Thus, research comparing MOSTs and non-MOSTs would complement this study and make a further contribution to developing a theory of productive use of student mathematical thinking.

Hiebert, Morris, Berk, and Jansen (2007) argued that teaching should be assessed based on how teachers make use of student responses in classrooms to foster understanding of mathematical ideas, rather than on the presence of recommended instructional features. Assessing teaching in this way requires a better understanding of both the nature of responses that could be used productively and the productive use of such responses. We see the attribute coding framework itself as tool for developing an assessment of the extent to which teachers are able to recognize and capitalize on MOSTs that vary with respect to their location within the mathematical or lesson terrain (*Context attributes*), as well as how the student thinking is expressed (*SM attributes*). Our data highlight that some attributes of student thinking that might cause teachers to pay less attention to them actually occur quite frequently within MOSTs. For example, one might anticipate that teachers would be fairly likely to notice incorrect thinking and feel compelled to respond to it in some way, yet the majority of MOSTs were found to be correct answers. Similarly, although teachers may intuitively notice MOSTs that are in the form of questions because, by their nature, questions presume a response, our data revealed that the majority of MOSTs are declarative statements, which may not naturally prompt a response. Although it is beyond our current analyses, studying the attributes of MOSTs in relation to assessments of teachers' responses to the MOSTs could provide insights into types of instances that may be easier or more difficult for teachers to recognize, informing professional development. Additionally, if it were found that teachers with different skill levels respond to MOSTs with various attributes in different ways, assessments based on these attributes could be developed to provide a more fine-grained means of assessing the practice of using student thinking than is currently available.

Research is also needed to identify and better understand professional development approaches that are effective in supporting teachers to use what is known about MOSTs to improve their classroom practice. Promising professional development approaches range from focusing teachers on anticipating MOSTs during their planning for a lesson, to coaching teachers in their classrooms to recognize MOSTs as they occur, to prompting teachers to reflect on MOSTs after a lesson. Approaches such as these could be investigated individually and in relation to each other to identify their affordances and constraints, thus informing professional development providers' decisions about the best use of the (typically limited) time and resources they have to support teachers to productively use MOSTs.

Conclusion

This study set out to contribute to our developing understanding of how to best support teachers' effective use of student mathematical thinking in their classrooms by investigating attributes of MOSTs—a high-potential subset of student thinking. The results provide insight into claims about the complexity of responding to students' mathematical thinking on the spot (e.g., Choppin, 2007; Jacobs et al., 2010). We now know that surface features of thinking, such as how it occurs, the form in which it is expressed, and how accurate it is, are not sufficient to determine whether the thinking should be pursued. Rather, responding effectively to student mathematical thinking requires careful attention to the content of the thinking to discern the underlying mathematical idea and what it might offer as the object of a class discussion. For example, some student questions may be best answered directly, but those that reflect a student's grappling with important mathematical ideas provide rich opportunities to engage the class in the type of mathematical activity advocated by current reforms (e.g. NCTM, 2014). Calculation and other surface mistakes may be dispensed with quickly, but students thinking that contains errors is often worth building on. Similarly, correct answers may be an indication to continue, or they may provide an opportunity to stop and engage the class in consolidating important mathematical understandings.

Despite the lack of easy answers about which thinking is worth building on in whole-class discussion, this work does provide some parameters that may make the process more manageable. For example, correct student thinking that does not involve sense making or multiple ideas or solutions is not likely to be worth pursing in a whole-class discussion. Being aware of patterns such as this one can help teachers avoid initiating unproductive discussions.

In general, this work supports the need for teachers to have criteria they can use for evaluating which student thinking is worth building on. The MOST Analytical Framework (Leatham et al., 2015) is one such set of criteria. Such criteria, in conjunction with the parameters contributed by this study, provide a starting place for designing teacher education to support teachers in developing the teaching practice of productively using student thinking.

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Appendix

	Prompt				Basis		Math Goal				Form			Accuracy					Intellectual Need			Type			
	SPT	OIS	OIP	TI	ITM	PRE	L	U	C	M	Q	T	D	COR	ICR	ICP	COM	N/A	OBV	TR	HID	I	SM	MIS	O
All MOSTs	**17**	**9**	**40**	**34**	**81**	**19**	**64**	**9**	**16**	**11**	**16**	**7**	**77**	**40**	**24**	**8**	**9**	**19**	**55**	**35**	**10**	**32**	**50**	**15**	**3**
Prompt																									
Spontaneous (SPT)					91	9	57	15	17	11	59	0	41	28	4	4	0	63	70	30	0	11	80	7	2
Open Invitation Selected (OIS)					63	37	59	4	30	7	11	22	67	41	30	4	11	15	56	37	7	33	44	19	4
Open Invitation Spontaneous (OIP)					98	2	66	10	14	10	4	4	92	37	34	13	10	6	55	33	12	39	36	21	4
Targeted Invitation (TI)					60	40	67	5	14	14	12	10	78	48	21	5	13	13	48	38	14	33	54	12	1
Basis																									
In-the-Moment (ITM)	19	8	49	25			62	11	19	8	20	8	72	37	25	9	7	22	56	35	9	31	53	13	3
Pre-Thought (PRE)	7	19	4	70			74	0	4	22	2	4	94	52	20	6	18	4	50	35	15	35	41	24	0
Math Goal																									
Lesson (L)	15	9	41	35	78	22					12	8	80	43	29	5	9	13	59	35	6	33	48	16	3
Unit (U)	29	4	46	21	100	0					46	4	50	21	4	17	13	46	83	17	0	13	75	13	0
Course (C)	18	18	36	29	94	6					23	4	73	29	24	9	9	29	44	36	20	36	49	13	2
Math (M)	17	3	37	43	63	37					7	10	83	50	13	13	10	13	27	46	27	37	43	17	3
Form																									
Questions (Q)	60	7	9	24	98	2	47	24	22	7				0	0	0	0	100	91	9	0	2	98	0	0
Tentative Statement (T)	0	30	25	45	90	10	70	5	10	15				30	45	5	20	0	40	50	10	55	40	0	5
Declarative Statement (D)	9	8	48	35	76	24	67	6	15	12				49	28	10	10	3	49	39	12	36	41	20	3
Accuracy																									
Correct (COR)	12	10	39	39	75	25	70	5	12	14	0	5	95						62	32	6	0	74	21	5
Incorrect (ICR)	3	12	56	29	84	16	76	1	16	6	0	13	87						38	51	10	88	1	10	0
Incomplete (ICP)	9	5	64	23	86	14	45	18	18	18	0	5	95						14	55	32	77	23	0	0
Combination (COM)	0	12	42	46	62	38	62	12	15	12	0	15	85						50	31	19	42	15	42	0
Not Applicable (N/A)	56	8	13	23	96	4	44	21	25	10	87	0	13						83	13	4	2	94	2	2
Intellectual Need																									
Obvious (OBV)	21	10	40	29	82	18	68	13	13	6	27	5	68	44	17	2	8	28				15	61	21	3
Translucent (TR)	14	10	38	37	80	20	65	4	16	14	4	10	86	36	36	12	8	7				52	41	6	1
Hidden (HID)	0	7	46	46	71	29	39	0	32	29	0	7	93	25	25	25	18	7				57	21	14	7
Type																									
Incorrect or Incomplete (I)	6	10	49	35	79	21	66	3	18	12	1	12	87	0	67	19	12	1	26	56	18				
Sense Making (SM)	26	9	29	36	84	16	61	13	16	10	31	6	63	58	1	4	3	35	67	29	4				
Multiple Ideas or Solutions (MIS)	7	12	55	26	69	31	67	7	14	12	0	0	100	55	17	0	26	2	76	14	10				
Others (O)	14	14	57	14	100	0	71	0	14	14	0	14	86	86	0	0	0	14	57	14	29				

Note: The sum for each category equals 100% horizontally, but not vertically.

Percentage of the Two x Two Interactions of the Seven Attribute Code Categories