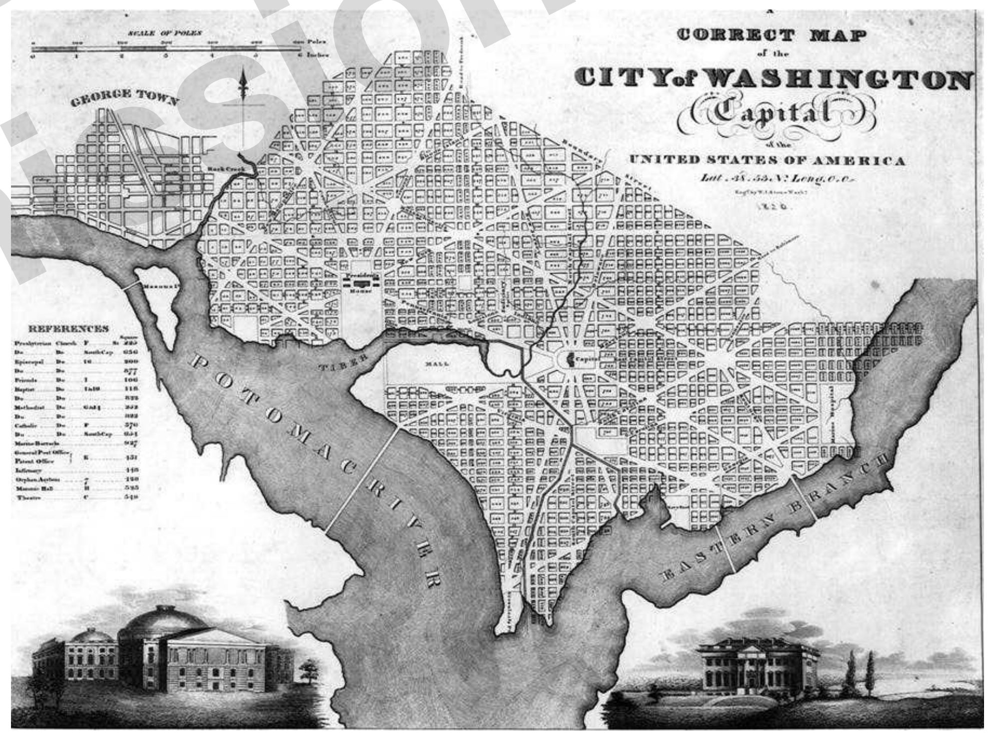
# Problematizing and Assessing Secondary Teachers' Ways of Thinking

Cuoco, Gates, Matsuura, Moore, Silverman, Stevens, & Sword





# Common Goals

## Continued Efforts to Understand “Knowledge for Teaching”

Understanding the complex relationship between teachers’ mathematical knowledge and their instructional practice (and student outcomes).

Not much debate that teachers need to know their content

Evidence is unclear about what exactly “know” means.

- Shulman: SMK, PK, PCK
- Ball: MKT
- Silverman/Thompson
- Cuoco

# Many Steams of Work…Moore & Silverman

Similar and/or normative responses may result from different mathematical meanings and this can an impact on the generativity and flexibility of individuals' understandings…

- Perceptual features (conventions, orientation of axes, shape of graph, "direction") inherent aspects of individuals' mathematical meanings;
- Mathematical (covariational) reasoning and abstraction at the core of individuals' mathematical meanings.
- Influence teachers' construction of key developmental understandings.

# Many Steams of Work…Cuoco, Matsuura, Stevens, & Sword

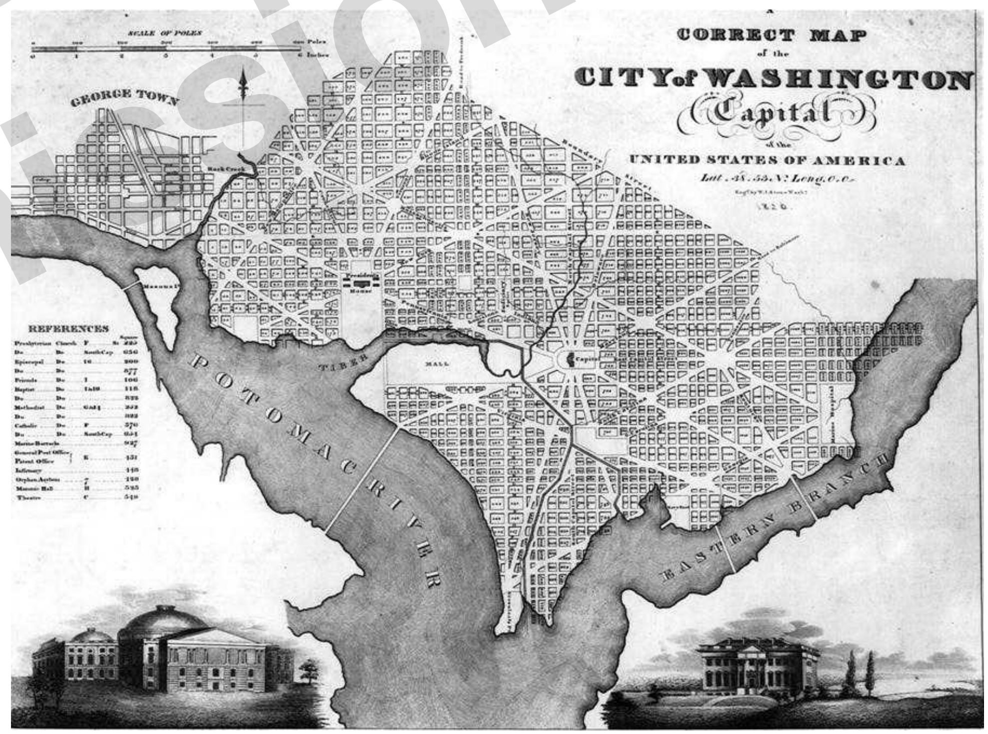
Mathematical Habits of Mind—the specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians— are valuable for teachers:

- They enrich and enhance the other ways of knowing mathematics;
- They can bring efficiency and coherence to teachers' mathematical thinking;
- They are embedded in national standards (like the Standards for Mathematical Practice).

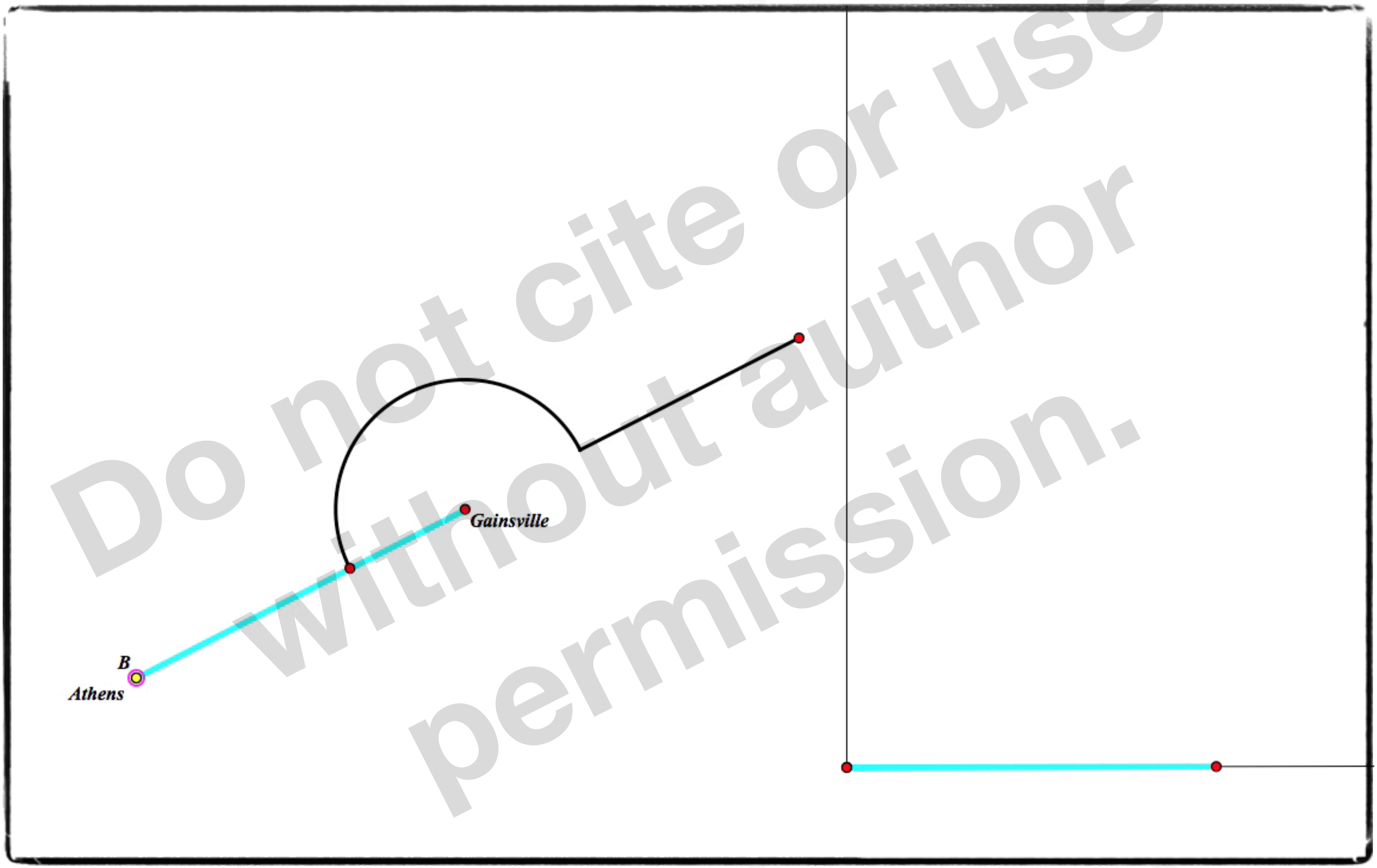
# Problematizing Prospective Secondary Teachers' Ways of Thinking

Kevin C. Moore and Jason Silverman






Car
Athens
Gainesville
Tampa


Gainsville
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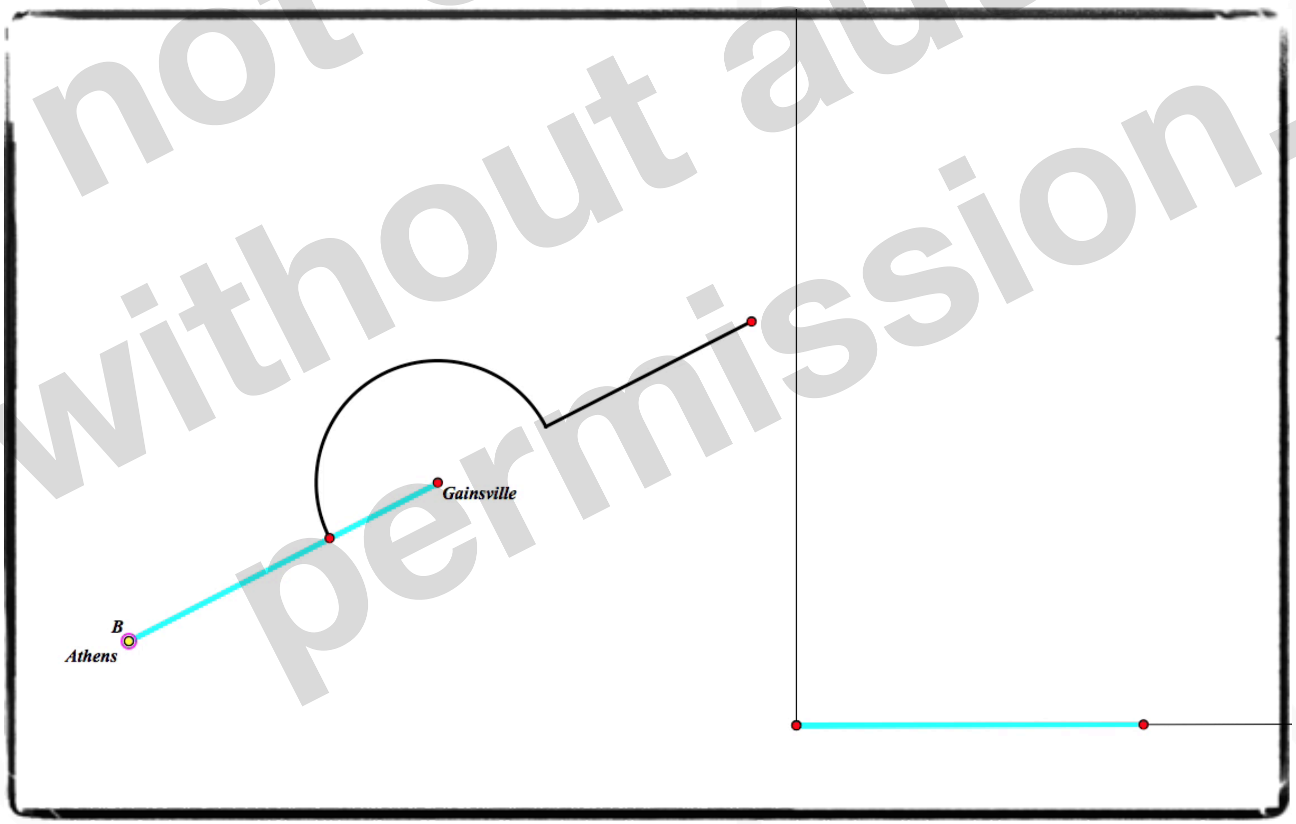
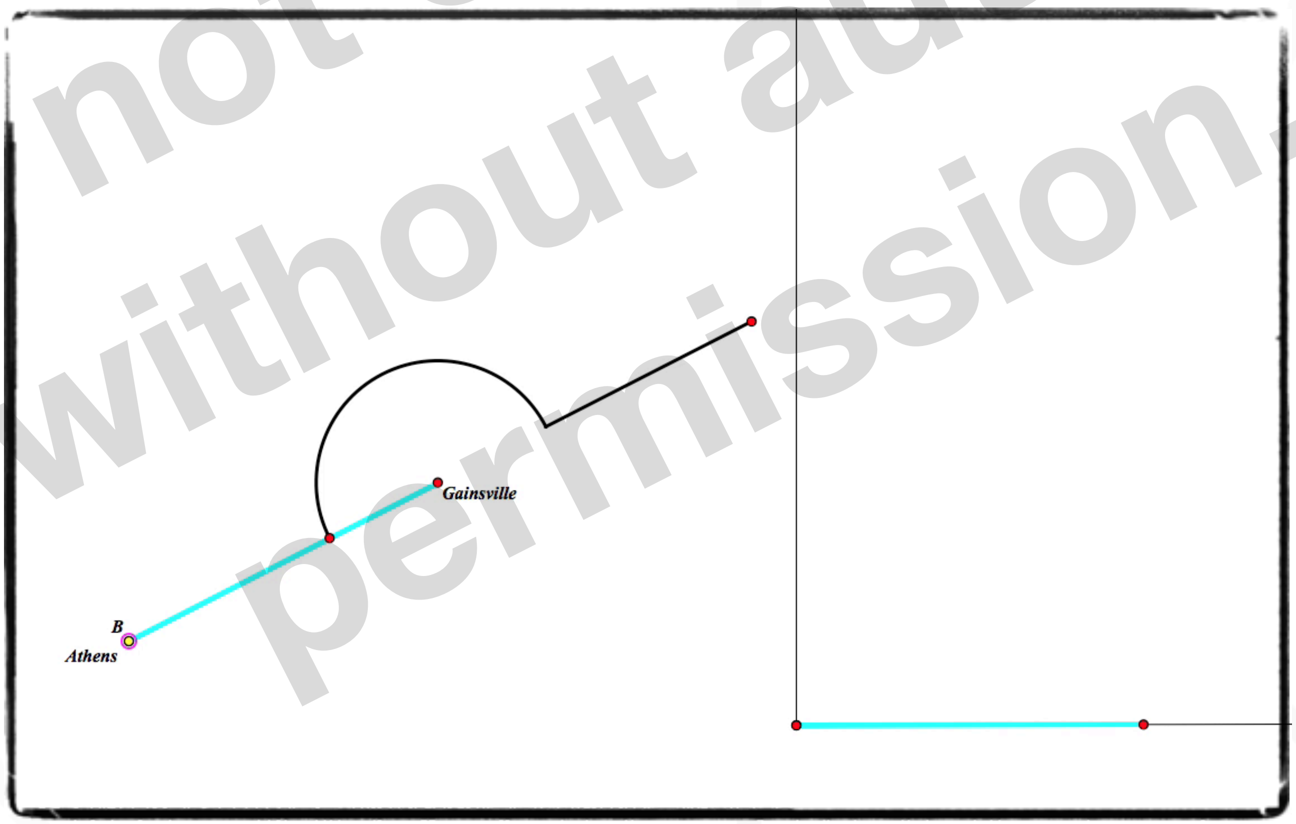
"**Covariational reasoning**…**coordinating two varying quantities** while attending to the ways in which they change in relation to each other."

Carlson et al., 2002



"…conceiving of a ***multiplicative object***—an object that is produced by uniting in mind two or more quantities simultaneously."

- Thompson, 2011



**Rate of Change and Accumulation** (Carlson et al., 2002; Johnson, 2015; Thompson, 1994; Thompson & Silverman, 2007; Zandieh, 2004)

**Limit** (Cottrill, Brown, & Dubinsky, 1998; Jacobs, 1999; Oehrtman, 2009)

**Function and Function Classes** (Ellis et al., 2015; Moore, 2014; Oehrtman, Carlson, & Thompson, 2008)

**Graphing** (Moore et al., 2013; Moore & Thompson, 2015)

Covariational reasoning is essentially **absent** from curricula and instruction.

Carlson et al., 2002; Moore, 2014; Smith & Thompson, 2008; Thompson, 2013

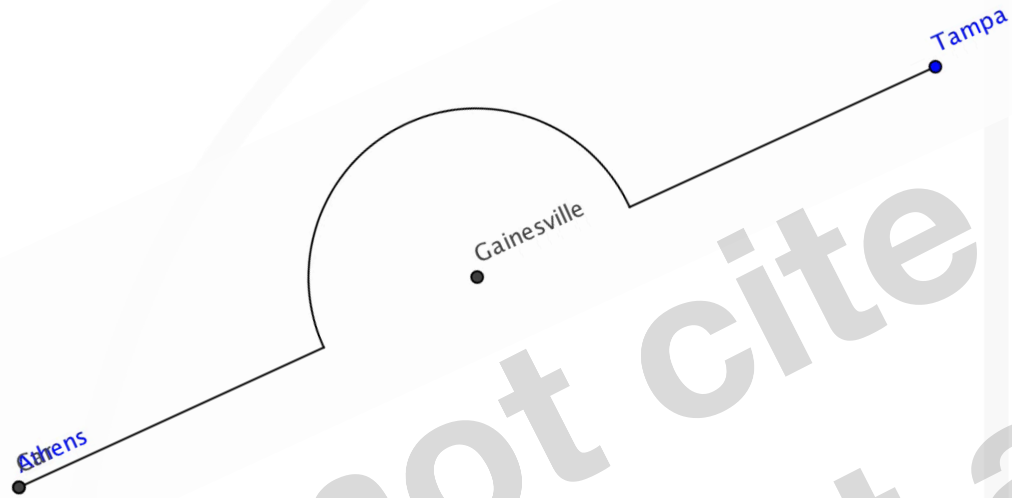
How do we work with teachers in ways that are sensitive to their established meanings, yet support shifts in their meanings that influence their practice?

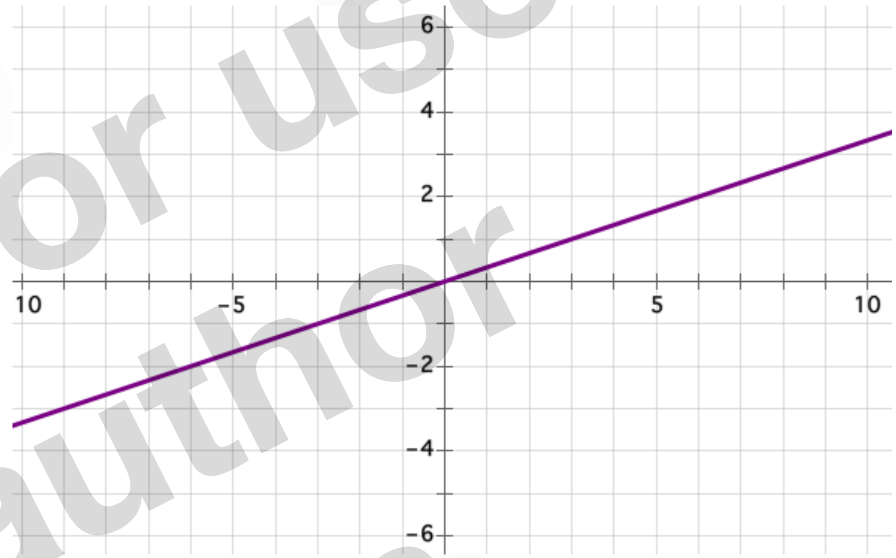
How do we work with teachers in ways that are sensitive to their **established meanings**, yet support **shifts** in their meanings that influence their practice?

Prospective secondary teachers during their first semester in a two-year program (juniors).

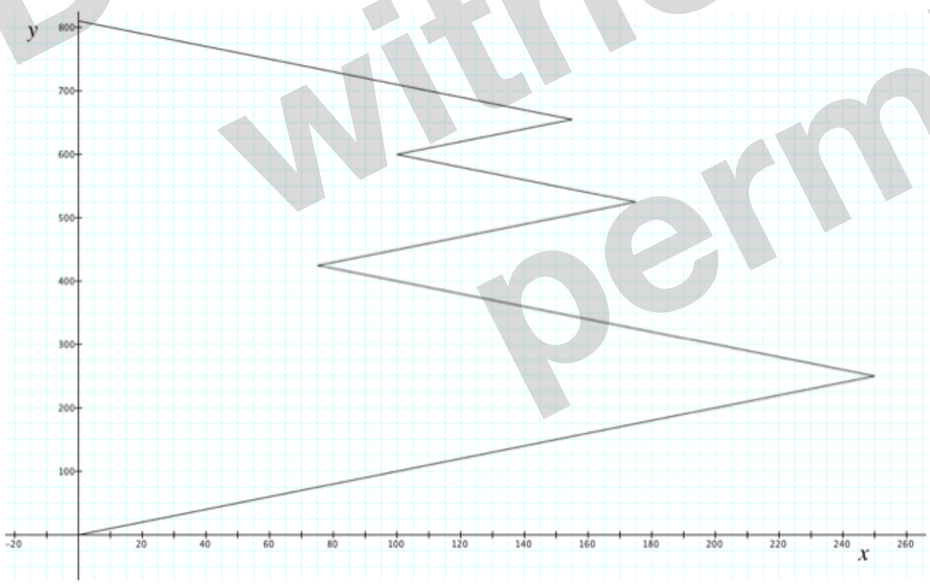
Clinical interviews (Ginsberg, 1997) and teaching experiments (Steffe & Thompson, 2000).

Ongoing and retrospective conceptual analyses (von Glasersfeld, 1995; Thompson, 2008).



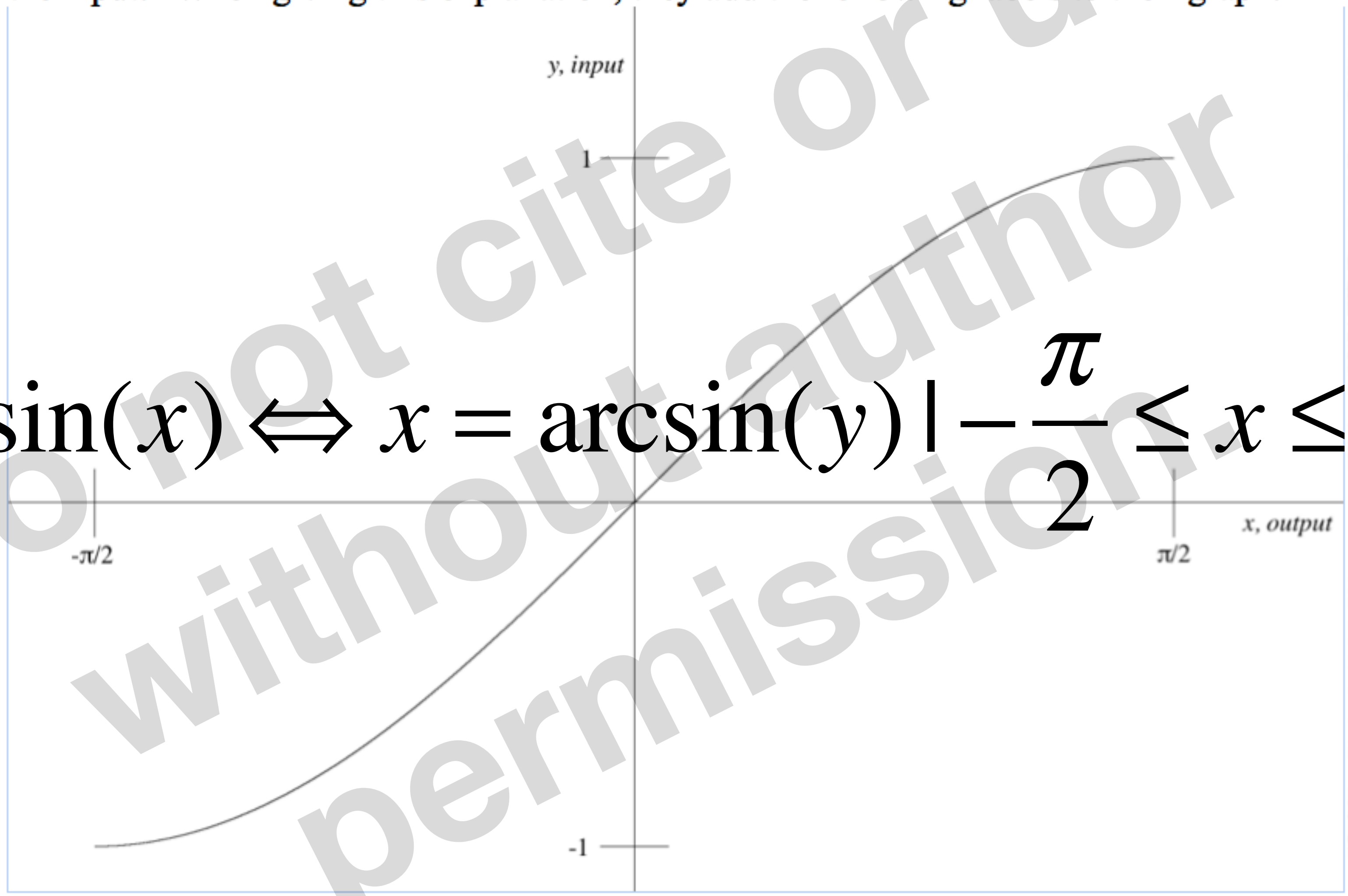

Is the following graph a function?




Thanks Desmos!

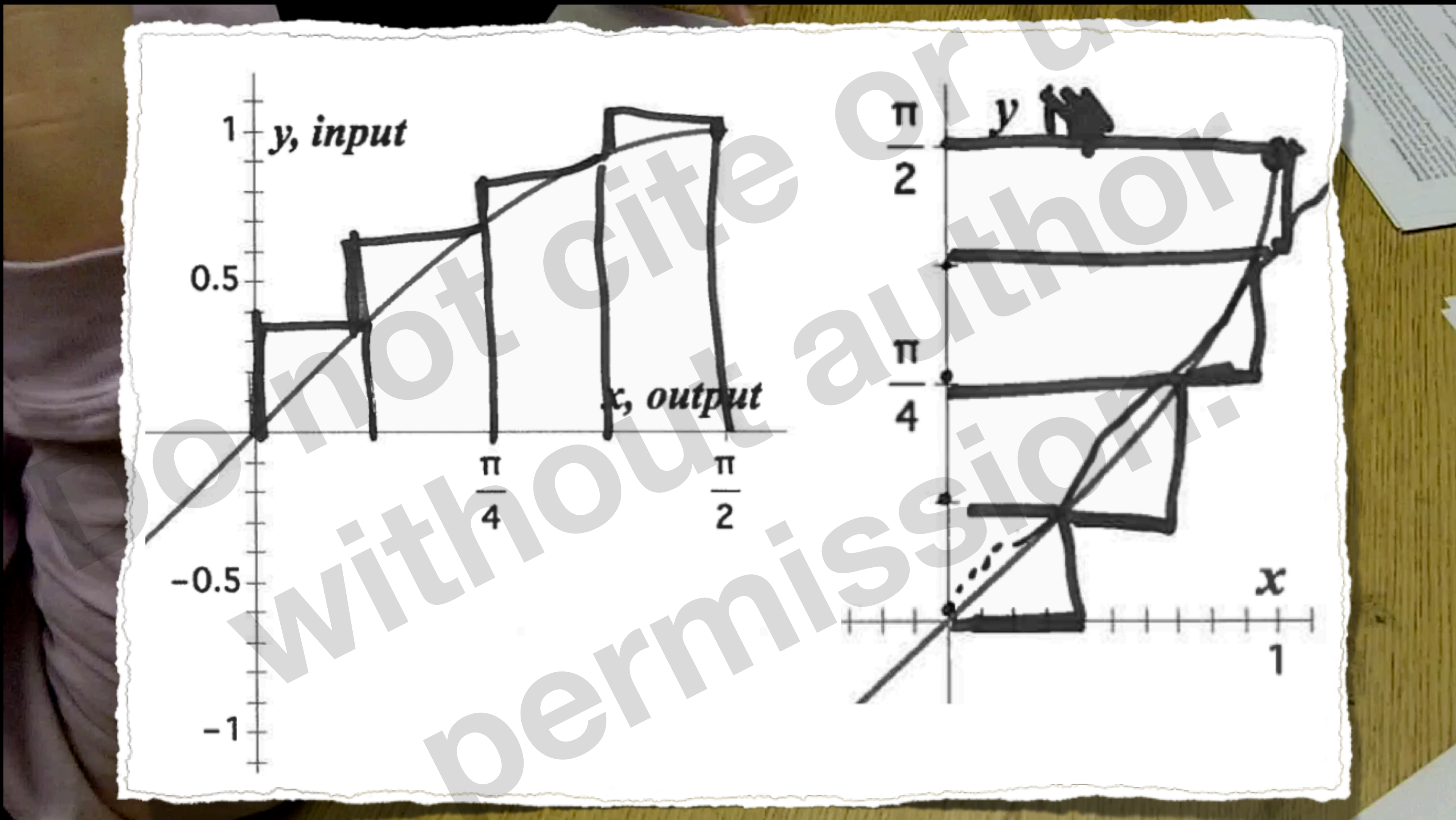
# Some Groundwork

When questioning the students from the previous problem, they claim, “Well, because we are graphing the inverse of the sine function, we just think about $x$ as the output and $y$ as the input.” When giving this explanation, they add the following labels to their graph.




$$y = \sin(x) \Leftrightarrow x = \arcsin(y) \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

What do you think about the students’ graph? Is it correct? Why or why not? How would you respond to the students?


1
y, input
0.5
x, output
−0.5
−1
π/4
π/2
y
π/2
π/4
x
1
Do not cite or use without author permission.

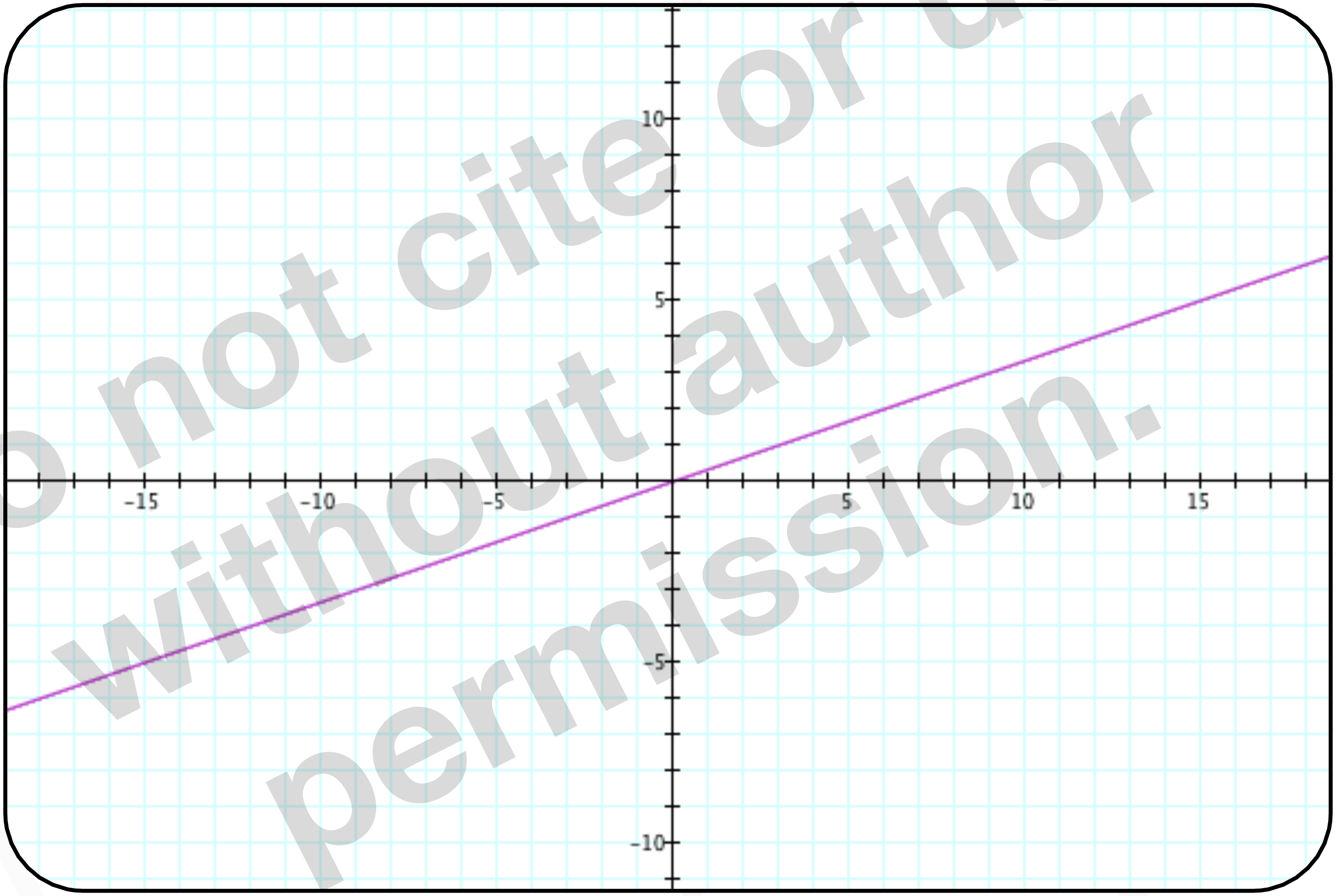
…plain sine graph…going to be different.

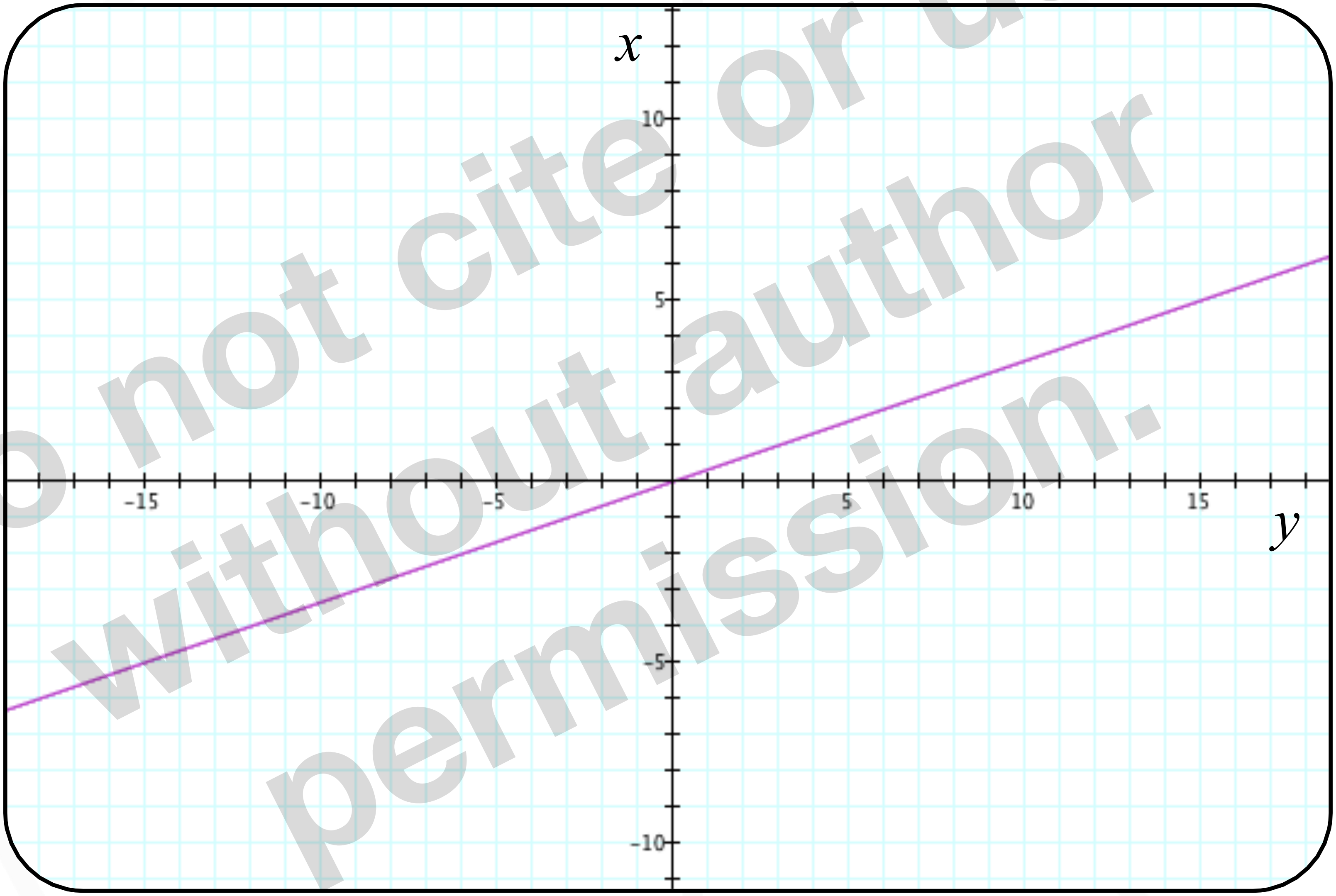


…it’s a graph everyone knows about.

Function names as pointers to interiorized covariational relationships.

Function names as one-to-one associations with shapes.


10
5
-5
-10
-15
-10
-5
5
10
15


x
10
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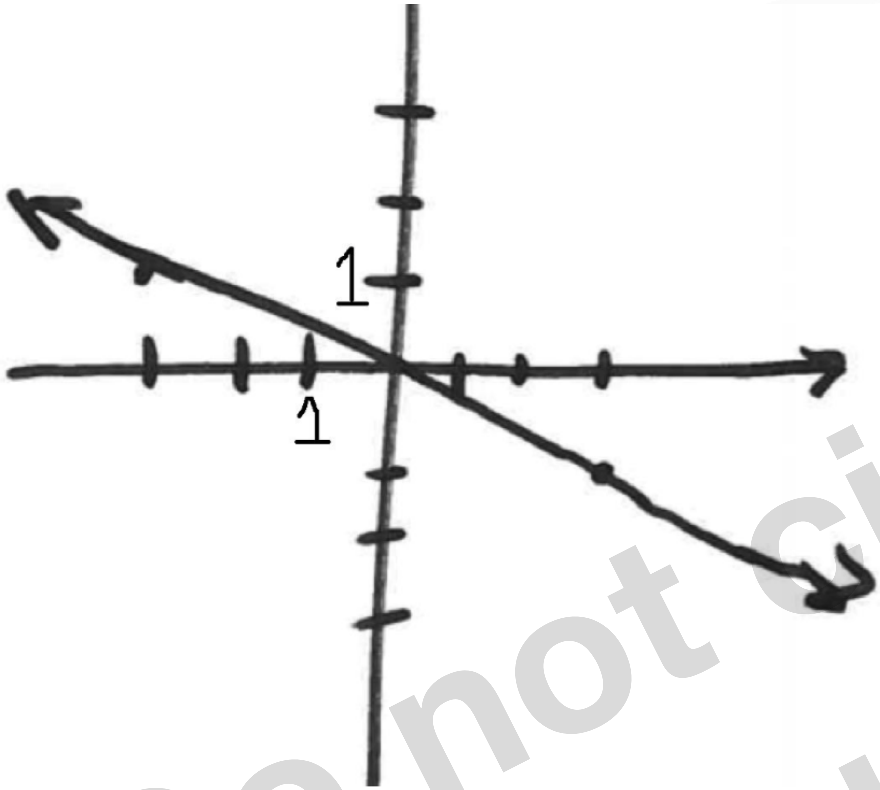
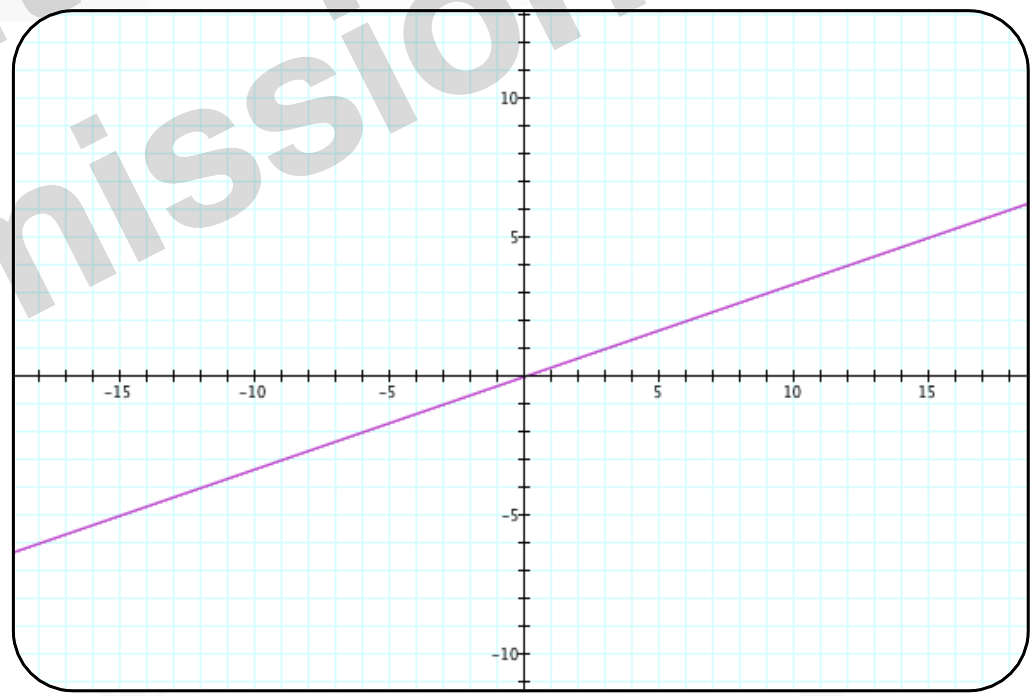
-5
-10
-15
-10
-5
5
10
15
y



1
1

10
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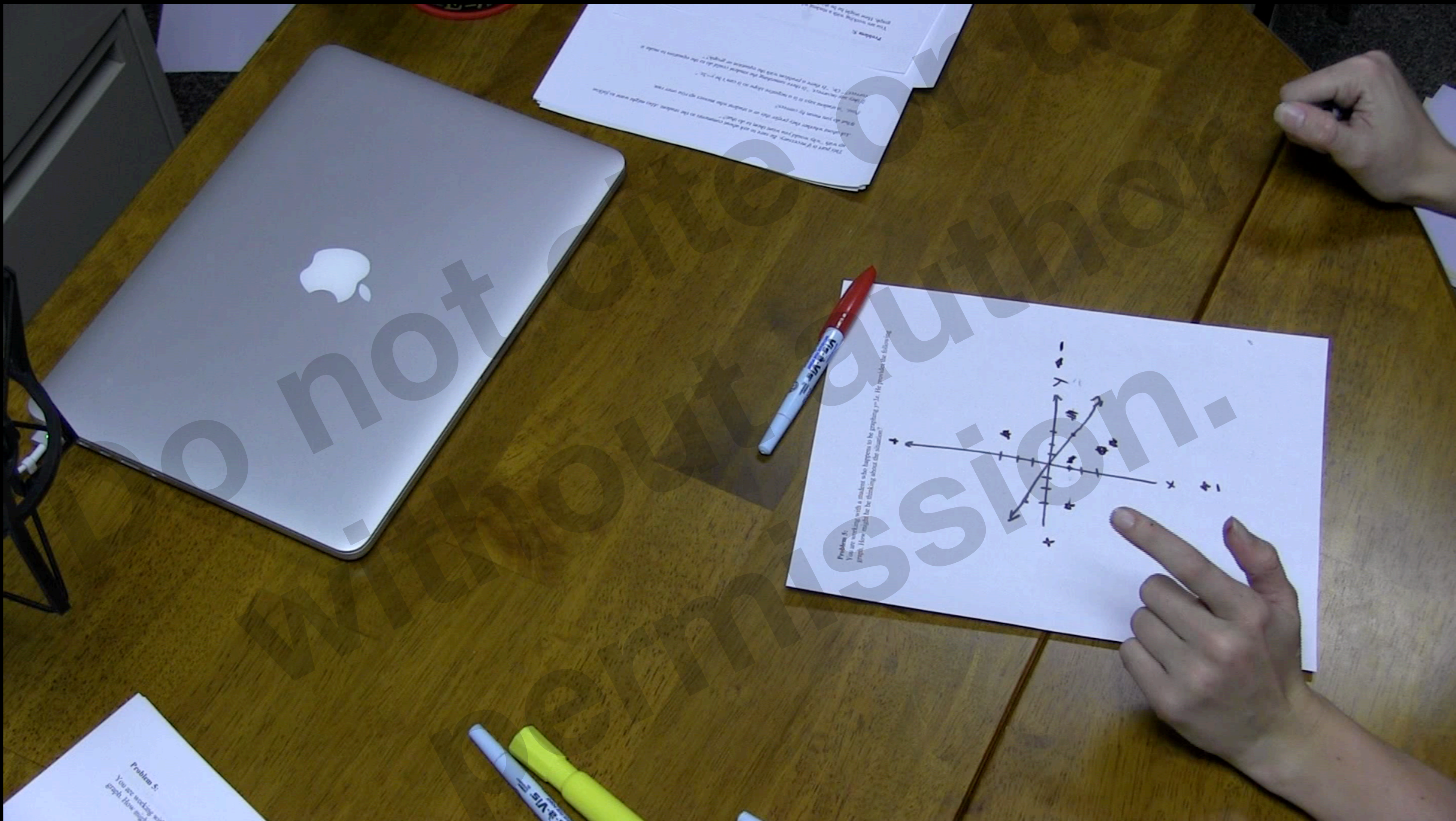
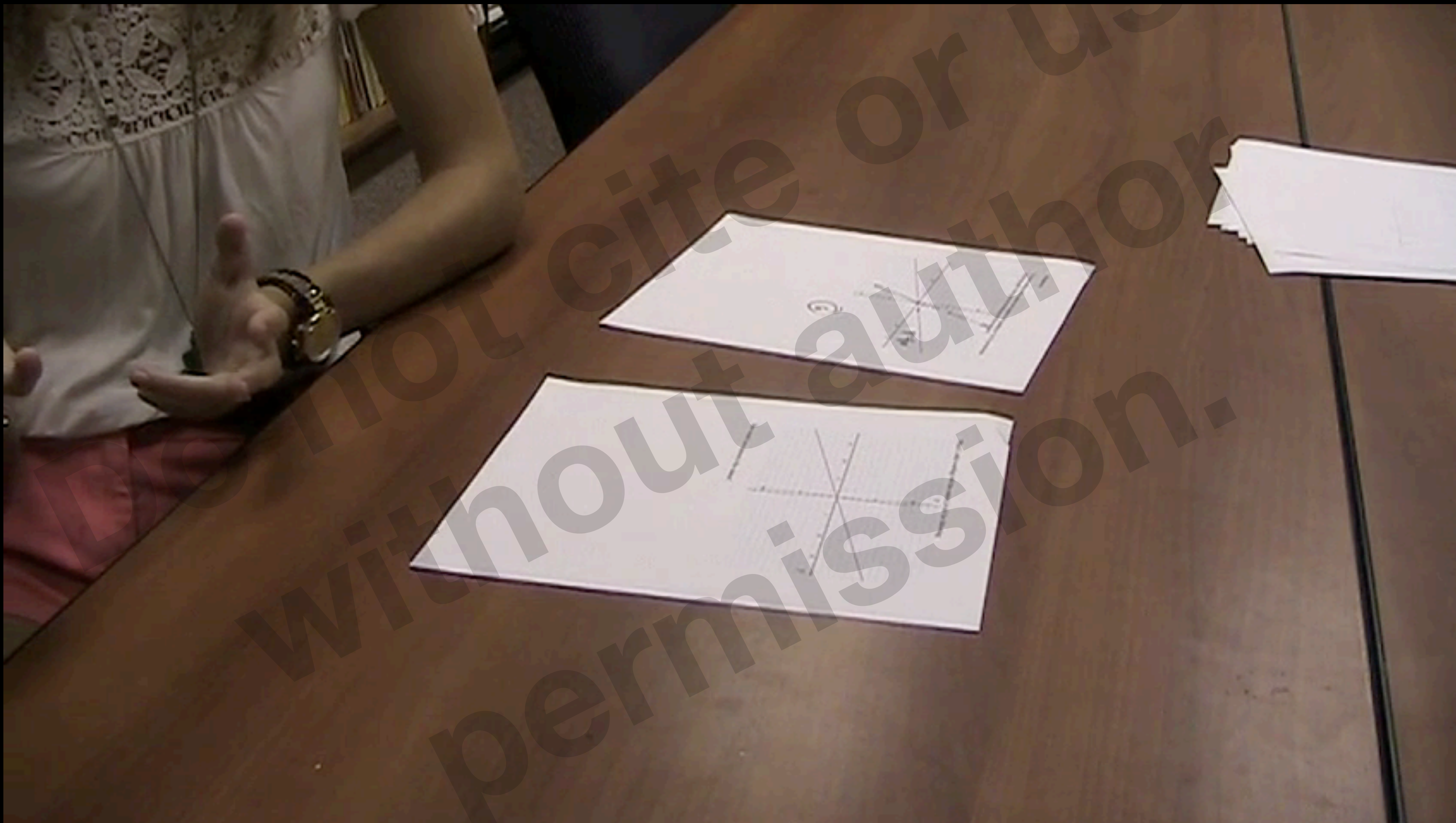
-10
-5
5
10
15

# But that's only visual…



…positive increase of one…positive increase of three.

# Where the slopes were…



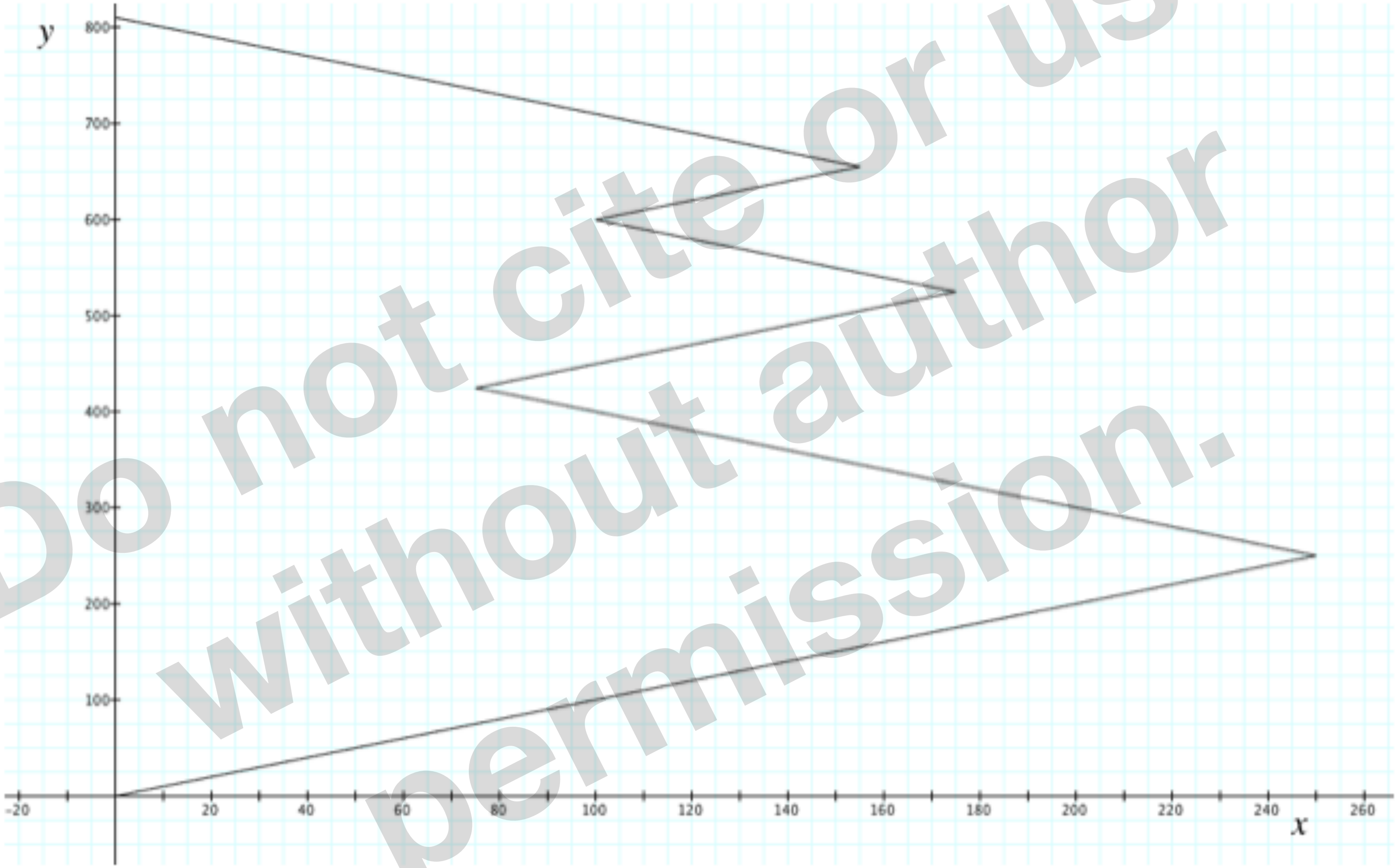


…which direction they're going.

Rate of change as an interiorized measure of coordinated quantities' values.

Rate of change (or slope) as an indicator or property of direction.

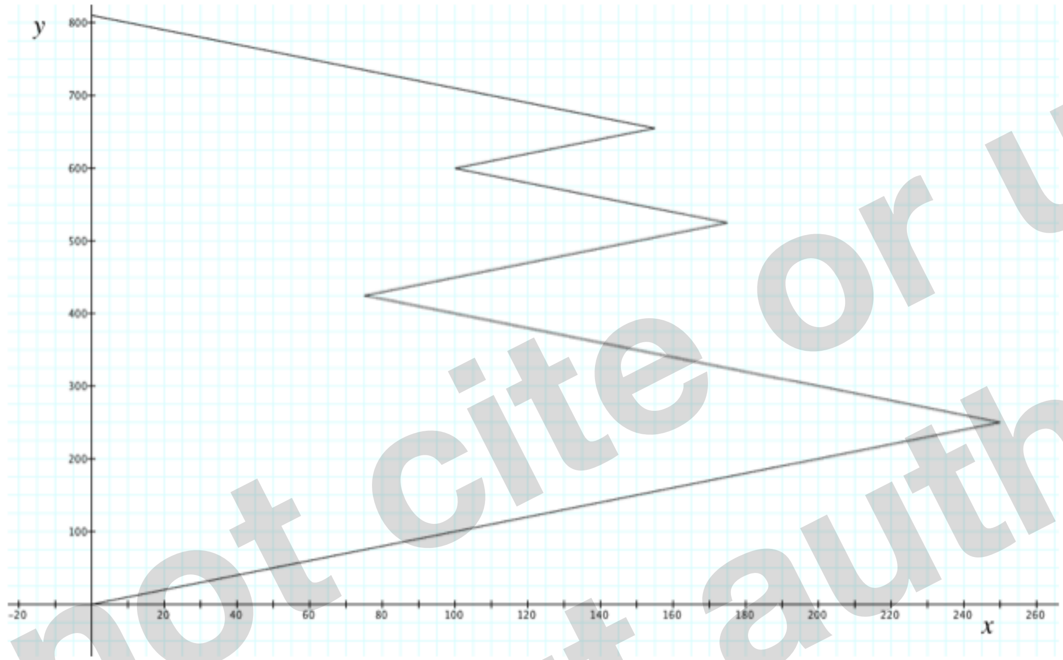
Is the following graph a function?



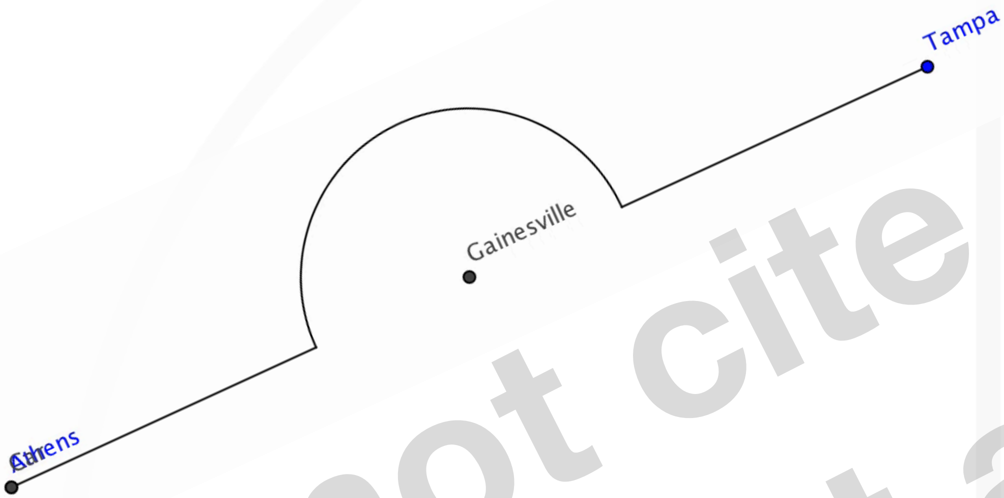


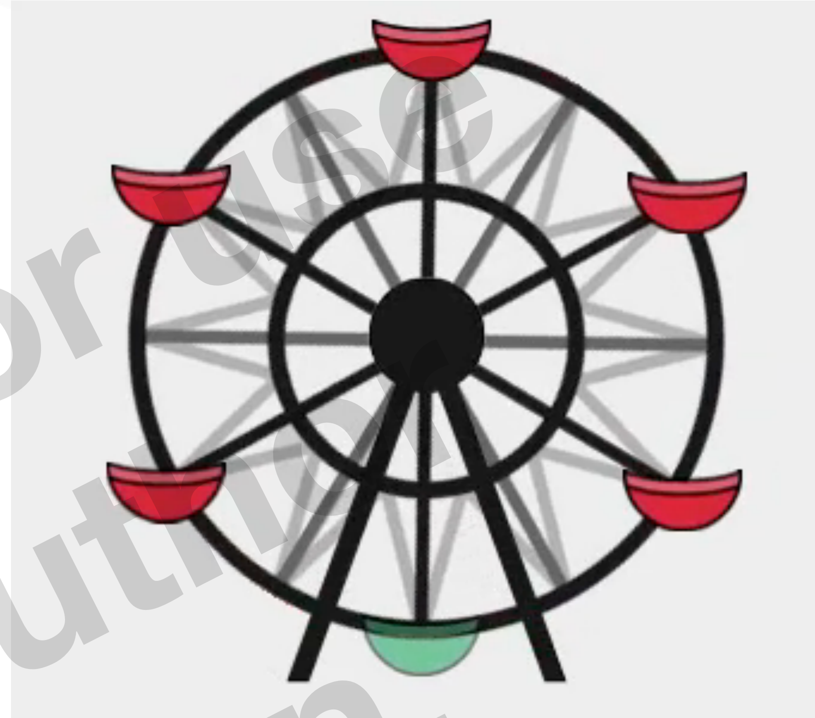

*A student claims…*

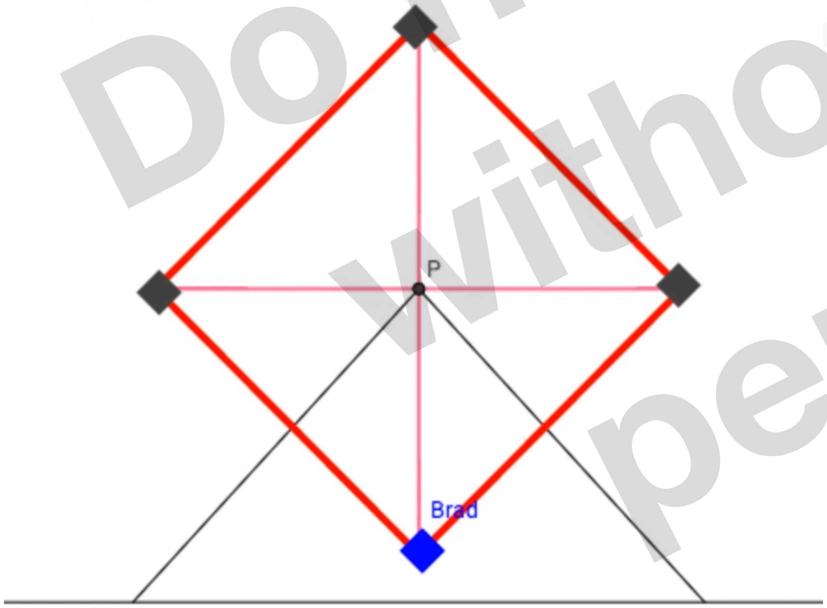
Is the following graph a function?

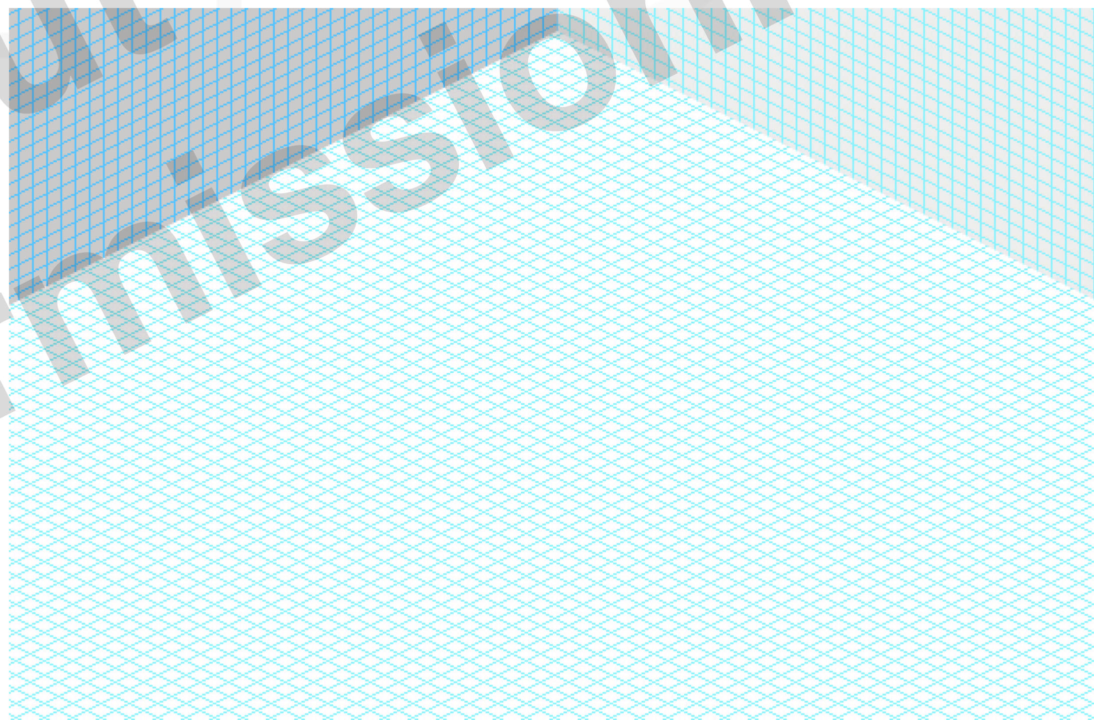


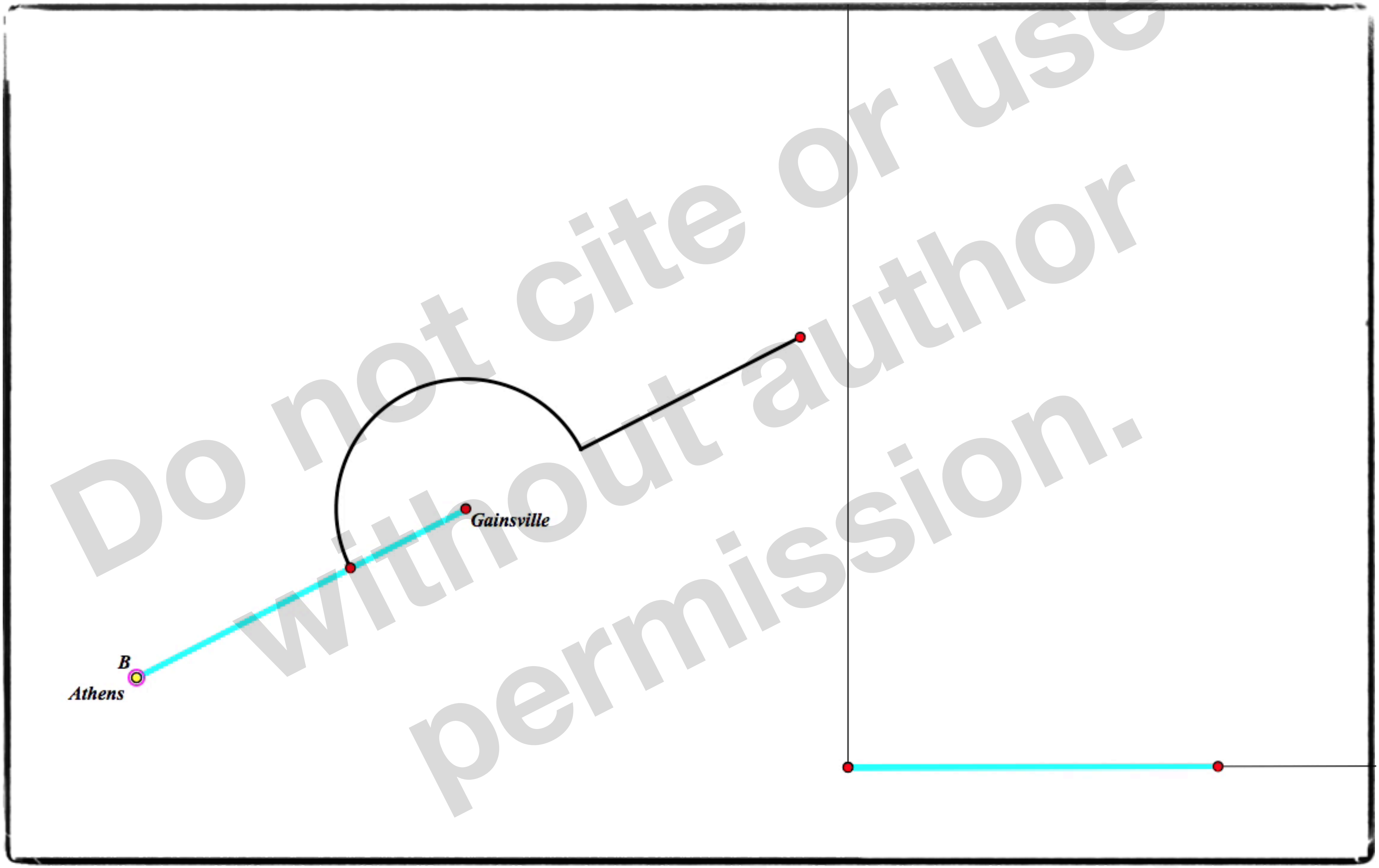

| Student Response Category | # out of 25 | Student Codes |
|---|---|---|
| Not true | 9 | S6, S7, S19, S20, S21, S23, S24, S26, S28 |
| True, if graph is rotated counterclockwise 90-degrees | 7 | S2, S4, S8, S14, S18, S27, S30 |
| True, if rotated counterclockwise 90-degrees and axes relabeled so that *y* and *x* were represented along the vertical and horizontal axes, respectively, in the new orientation | 1 | S25 |
| True | 7 | S1, S5, S10, S13, S16, S22, S29 |
| Unsure | 1 | S3 |


Tampa
Gainesville
Athens




P
Brad

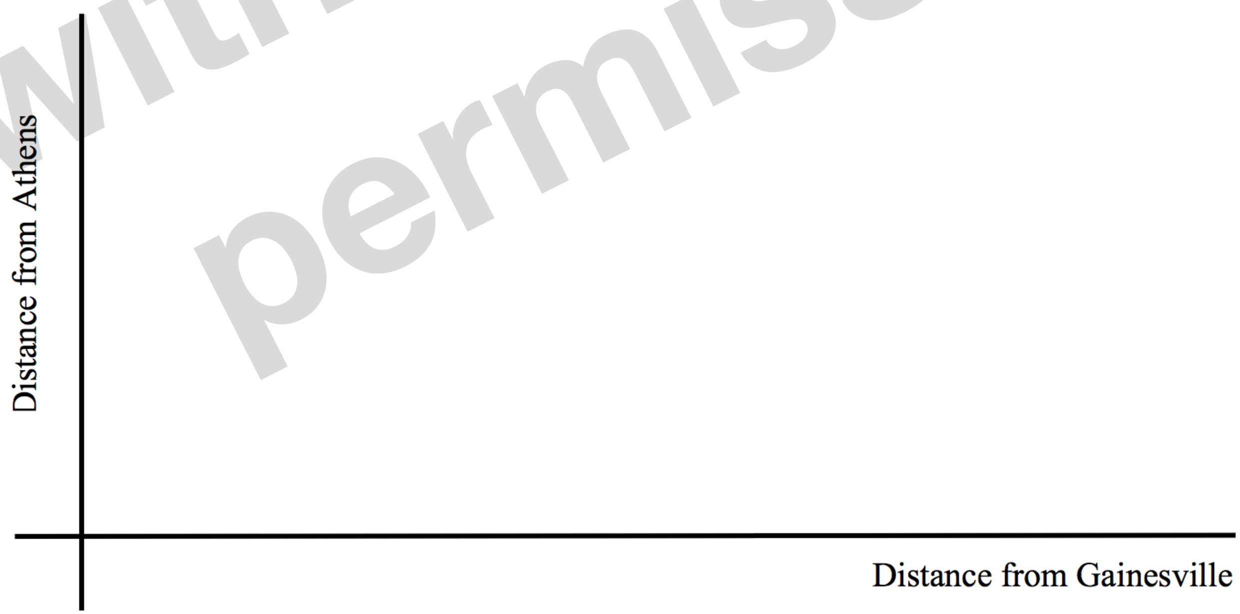


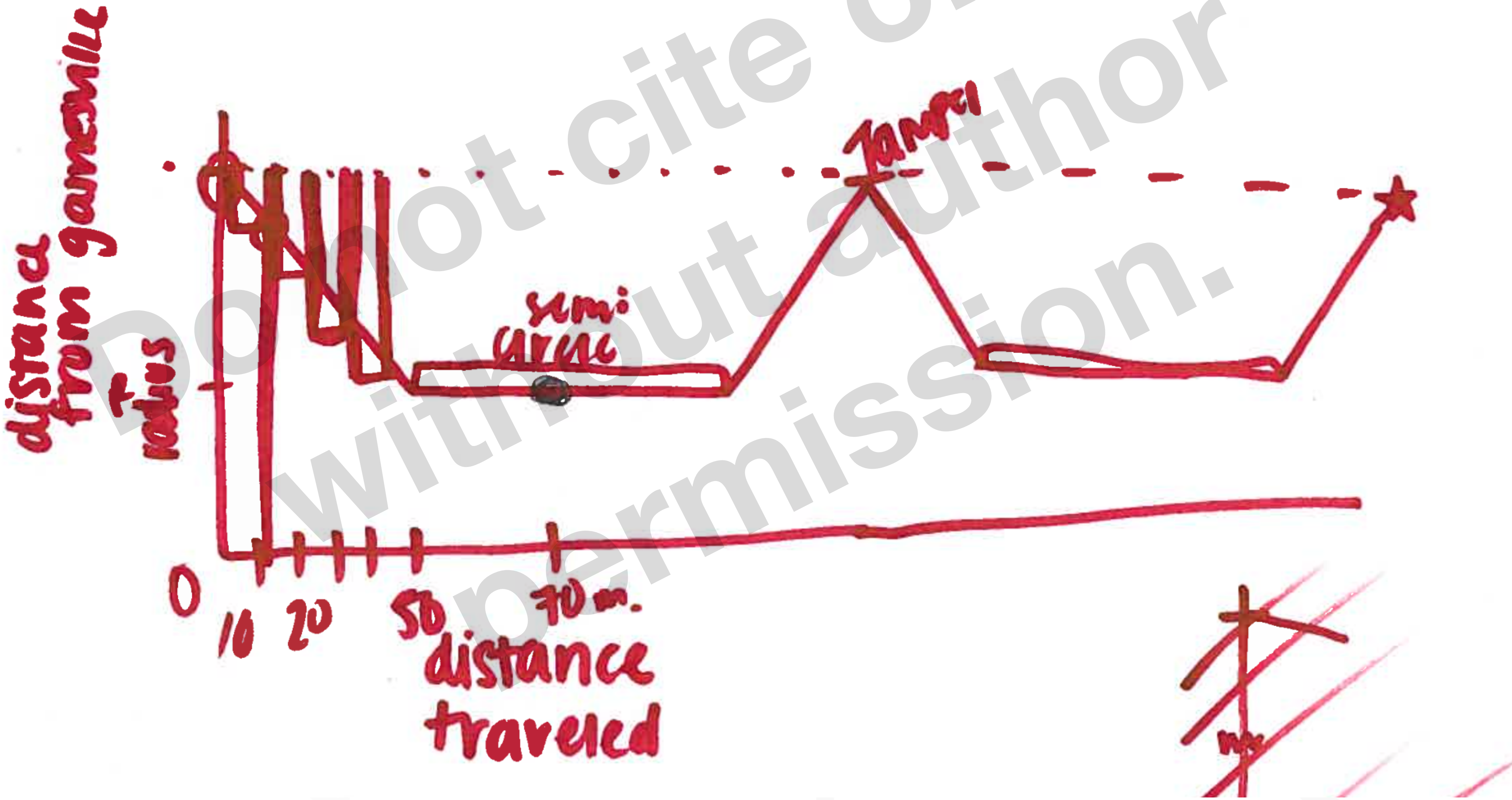

Gainsville
B
Athens

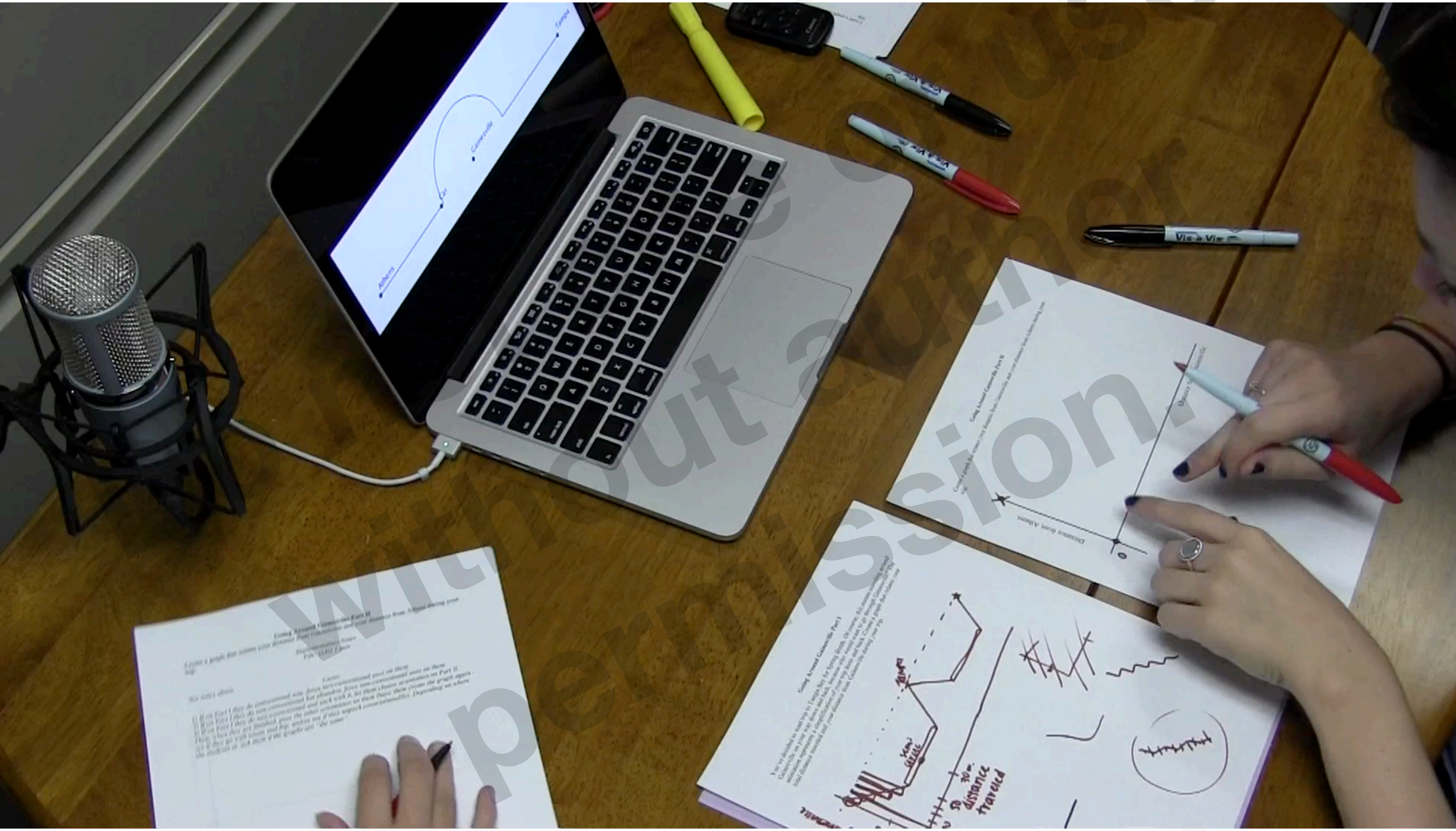
# An Illustration

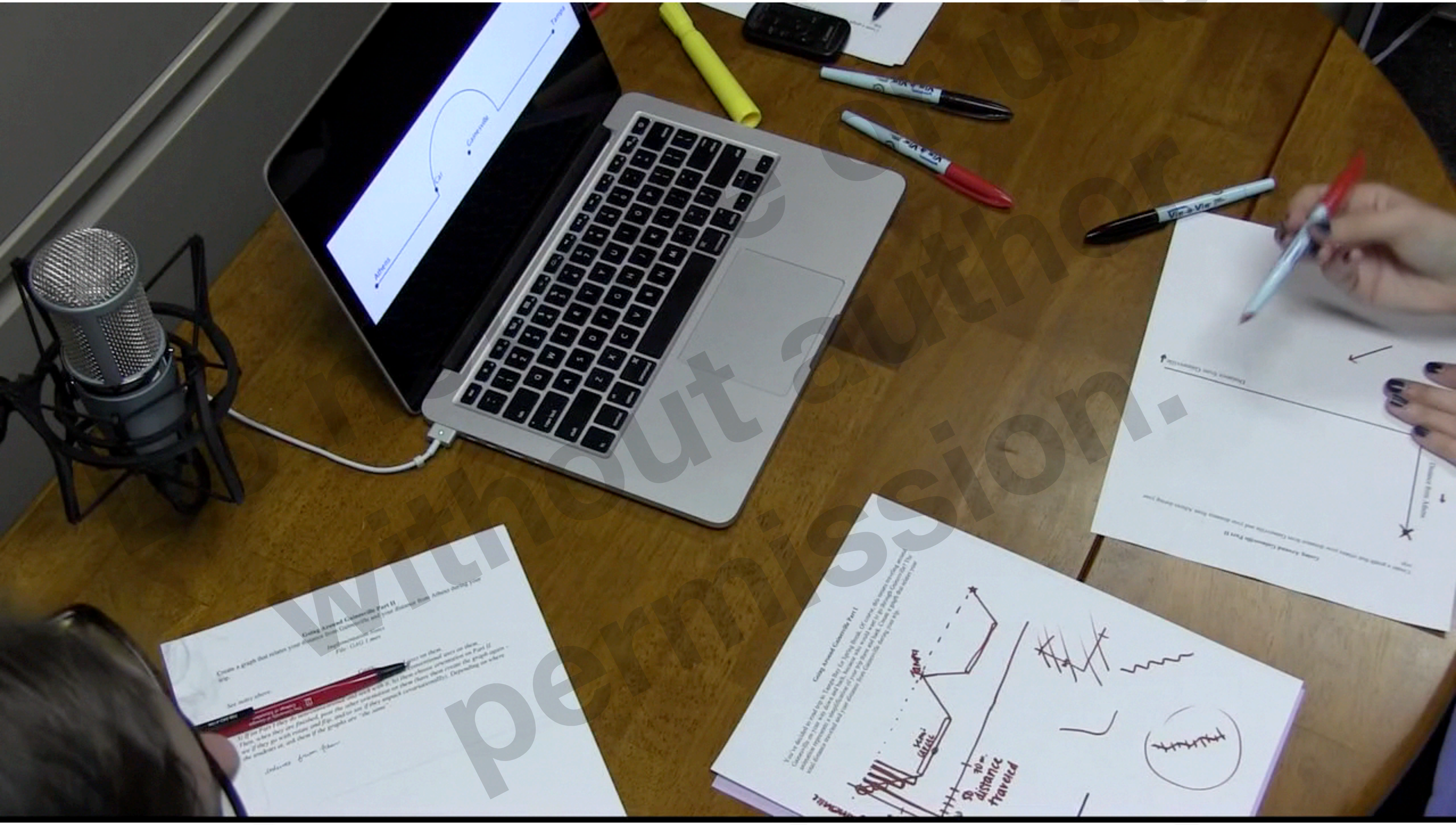
# “It’s backwards.”

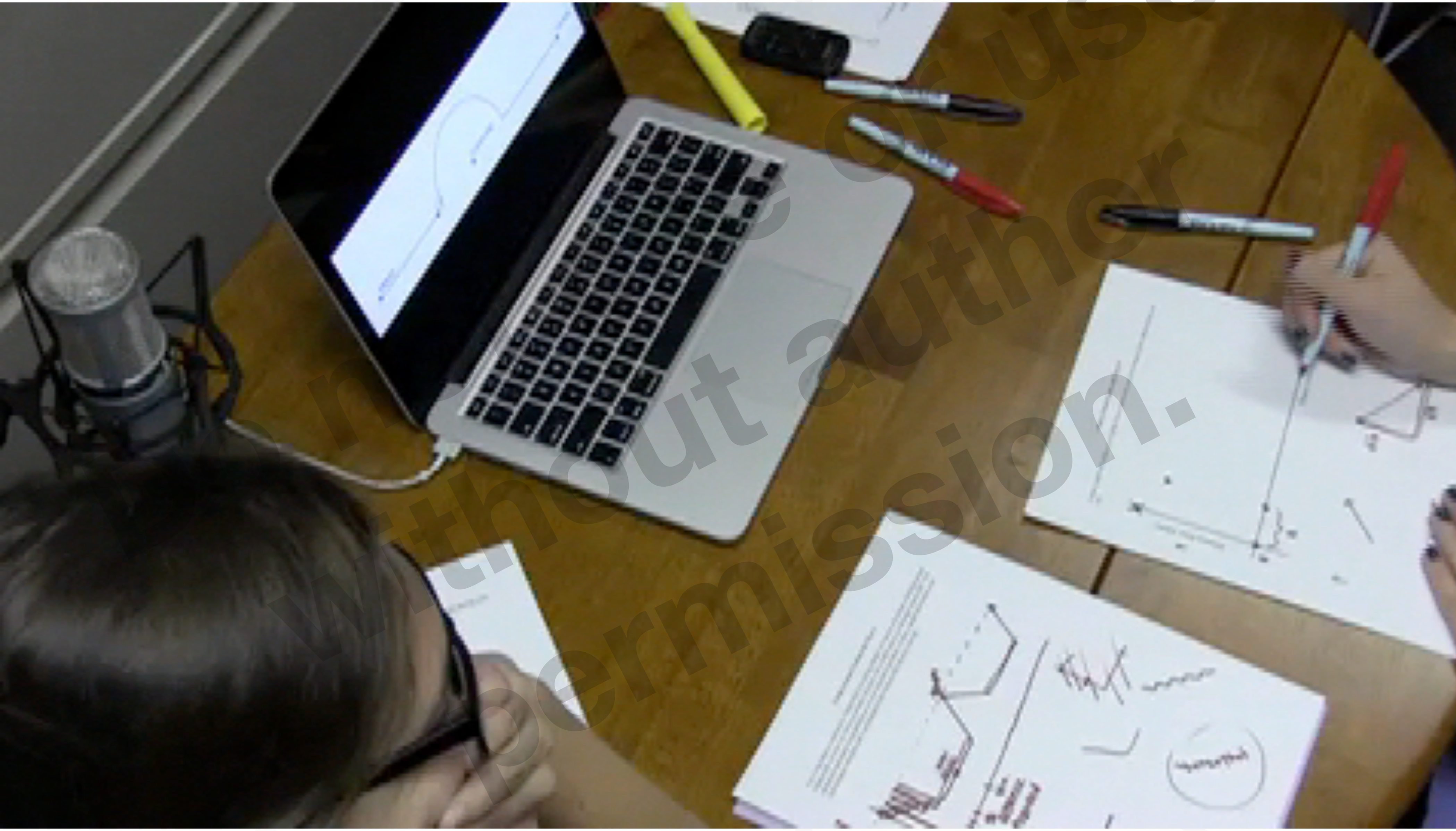

Create a graph that relates your distance from Gainesville and your distance from Athens during your trip.




distance from gainesville
radius
semi circle
Tampa
0
10
20
50
70 m.
distance traveled


Athens
Gainesville
Tampa
Distance from Athens





Learning

Discussion

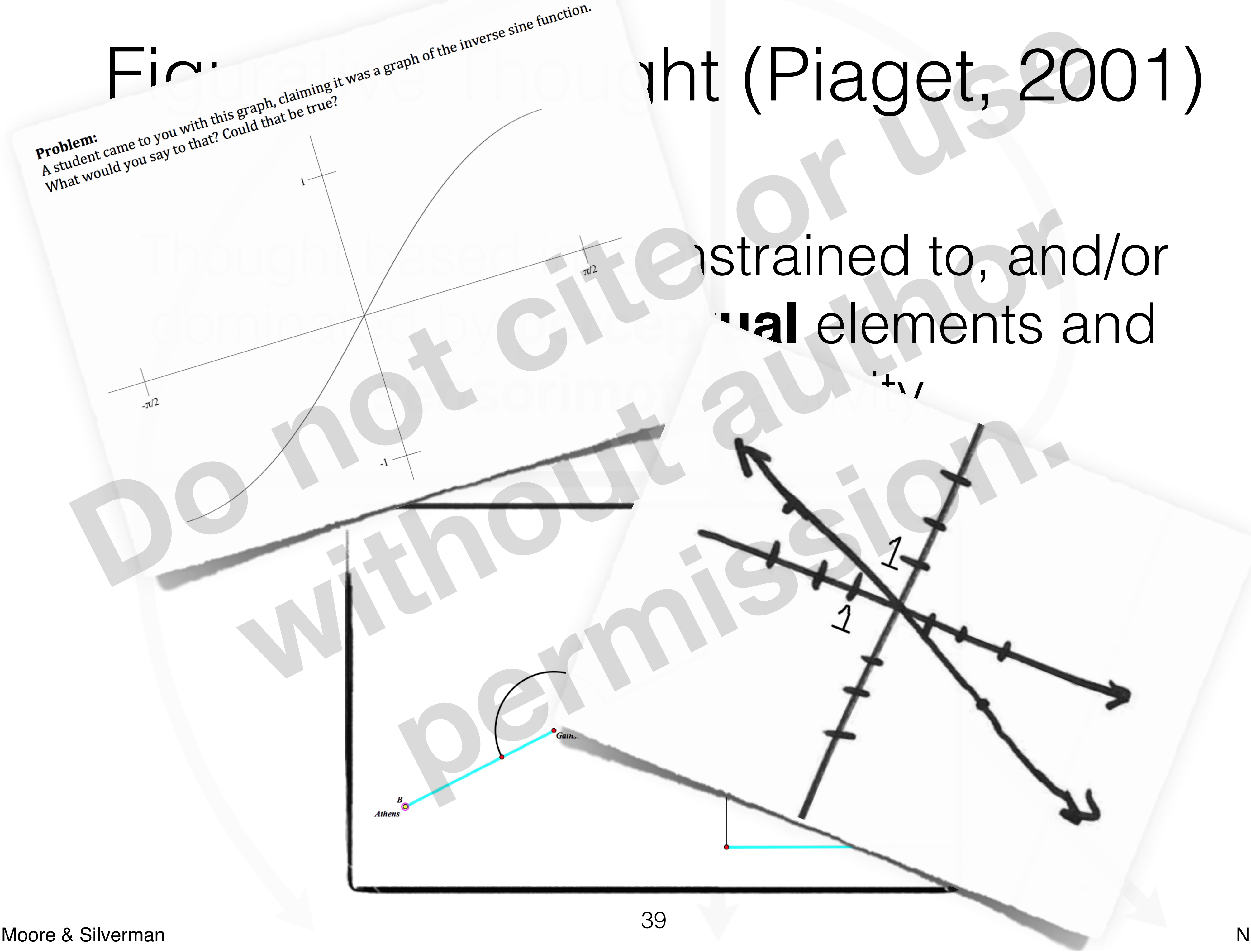
Teaching

# Learning

Students are establishing ways of thinking about graphs (and associated topics) that are tied to **figurative results and particulars** of their activity.

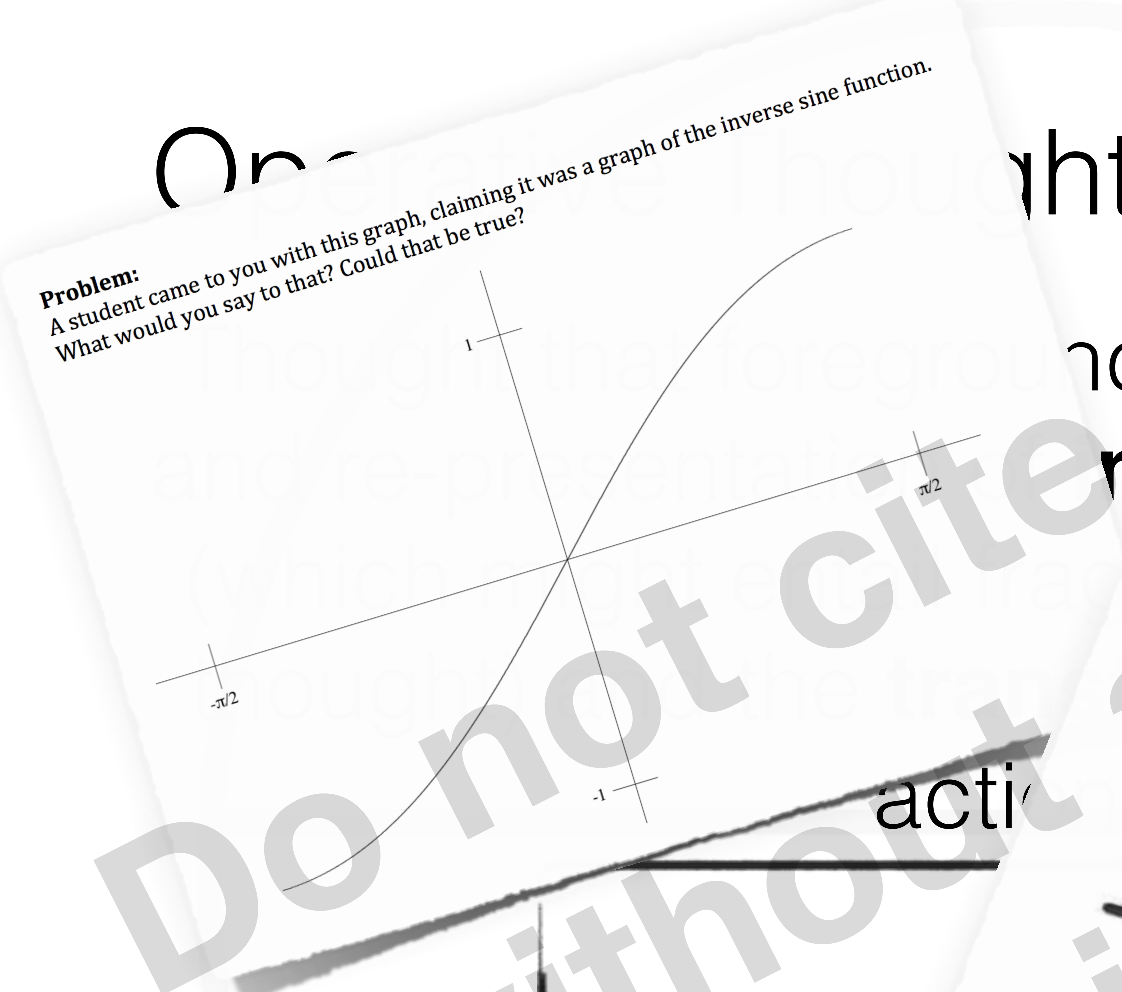
# Figurative Thought (Piaget, 2001)

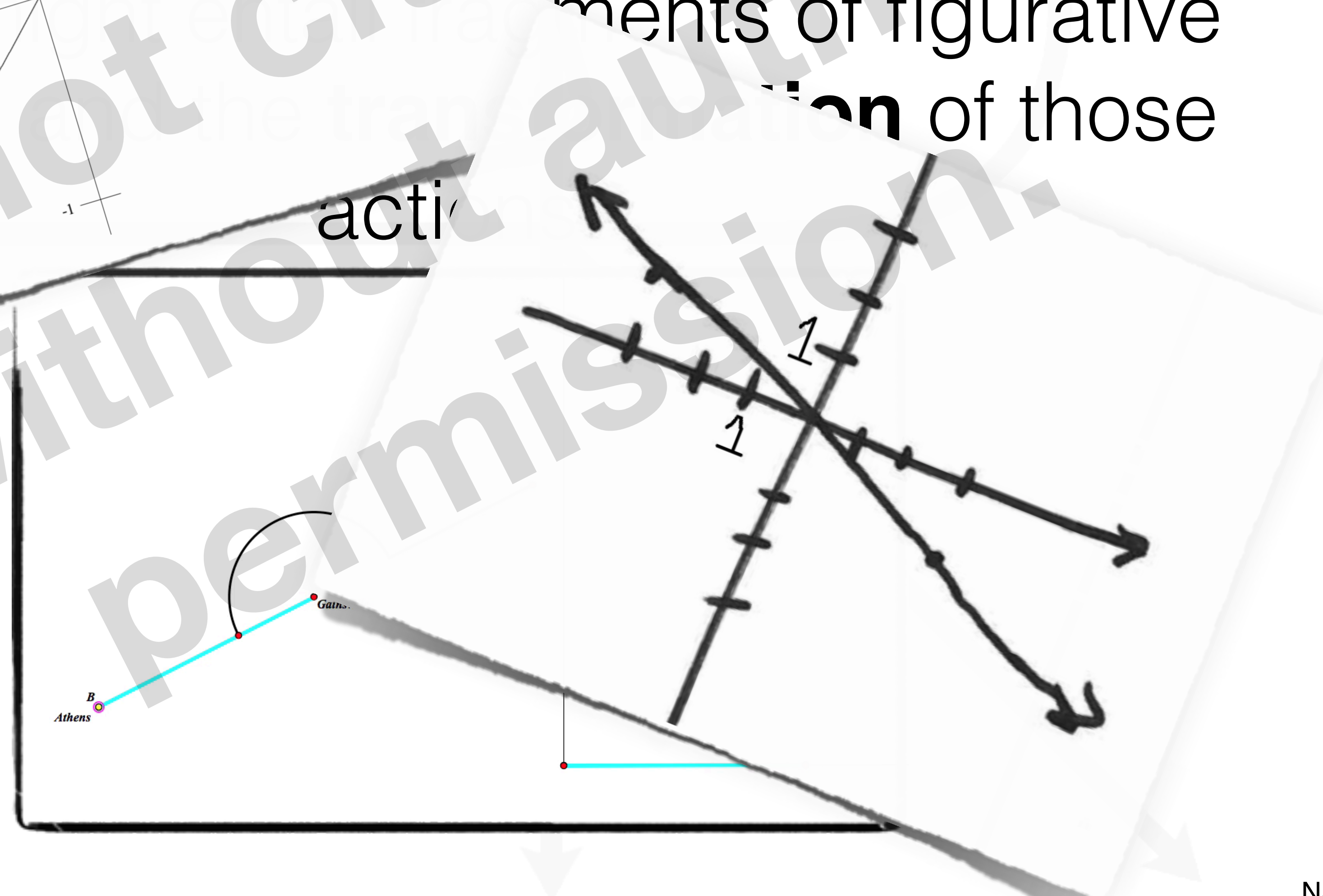
- Thought based in, constrained to, and/or dominated by perceptual elements and sensorimotor activity



# Operative Thought (Piaget, 2001)

Thought that foregrounds the coordination and re-presentation of **internalized actions** (which might entail fragments of figurative thought) and the **transformation** of those actions



Learning

Discussion

Teaching

This is not happening by accident…

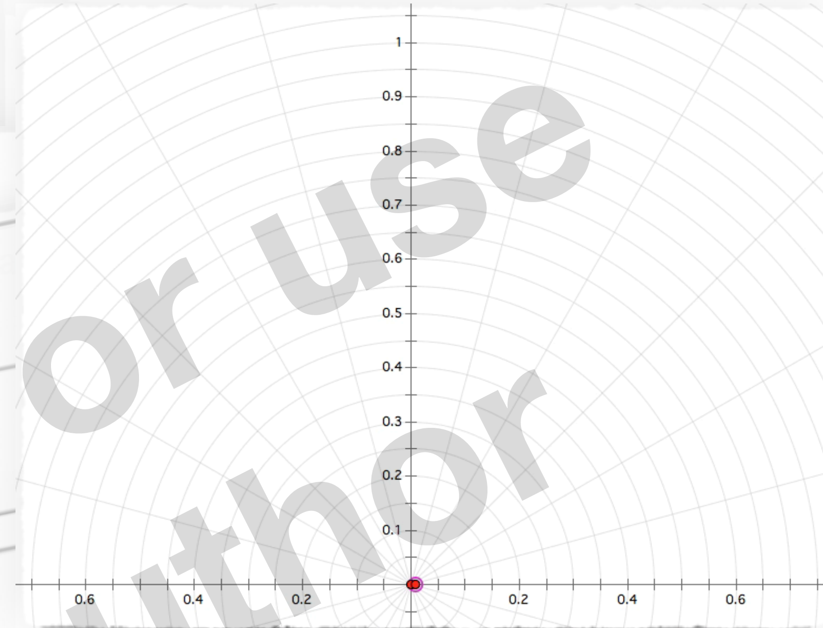
# Teaching

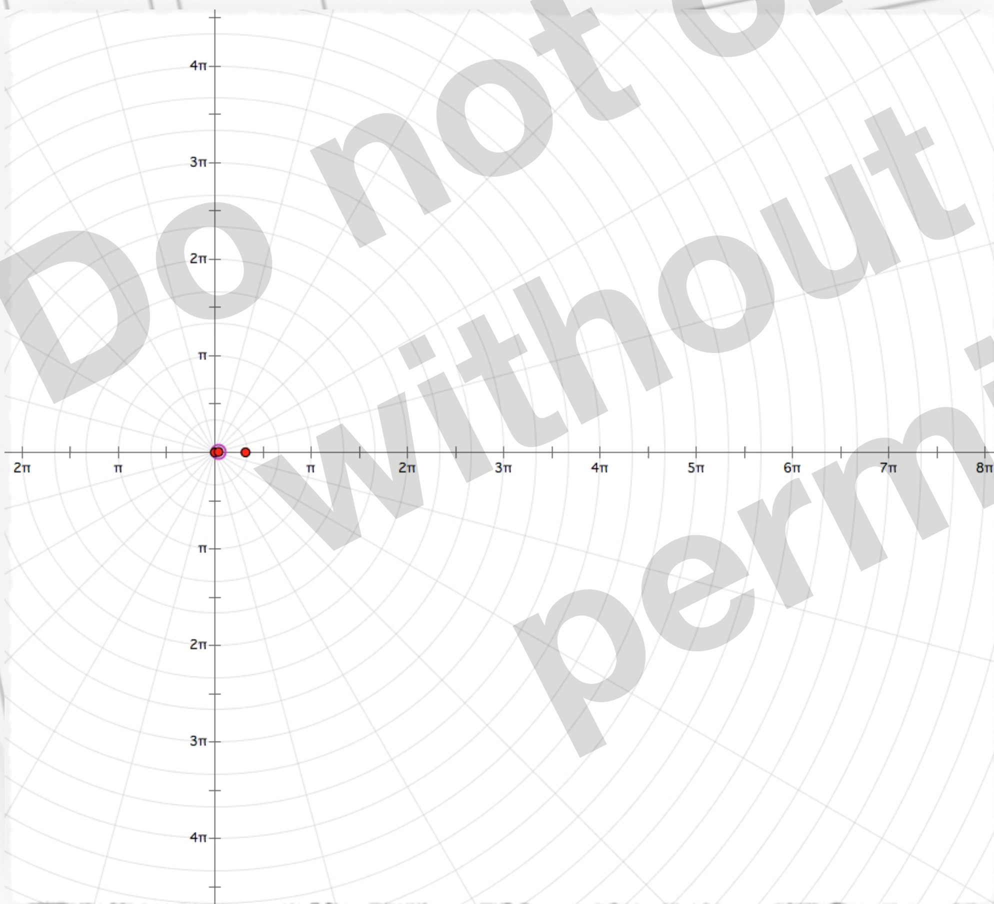
Students and teachers are not having opportunities to construct and differentiate between actions and operations (that we perceive to be) critical to an idea and those that are merely a convention of a representation of that idea.

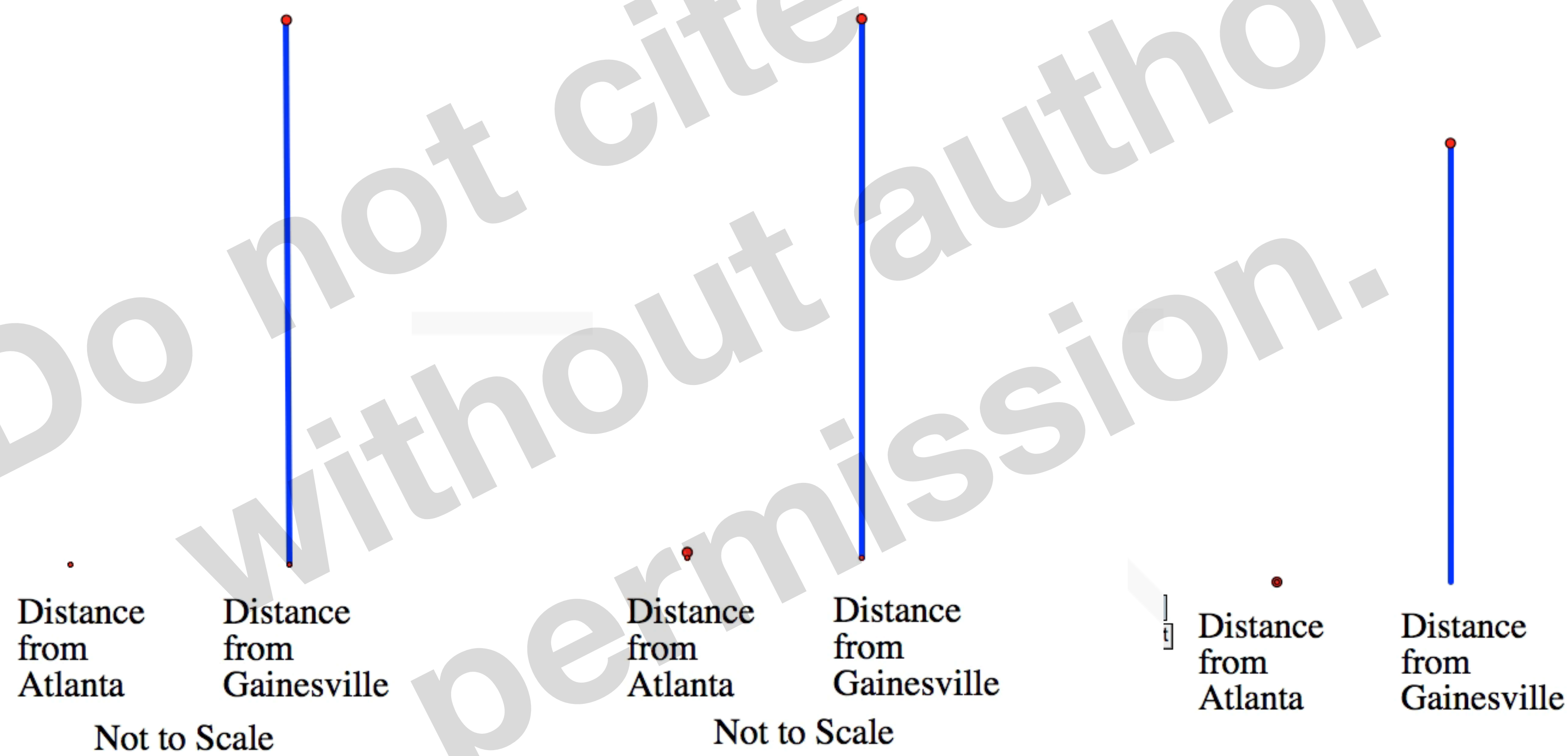
# What next???

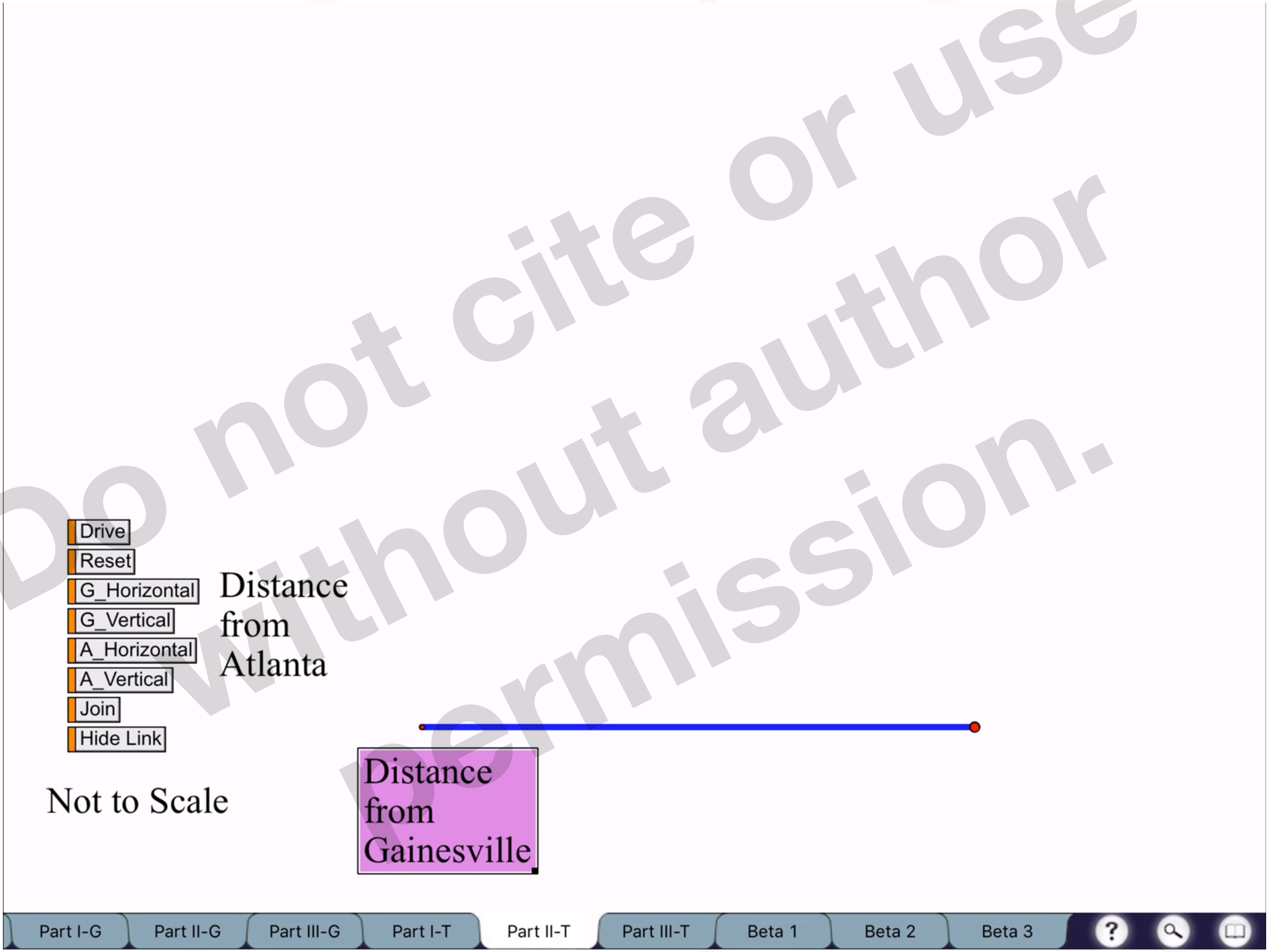
**What is key to a concept or idea and what is a convention of a representational system for conveying that concept or idea?**

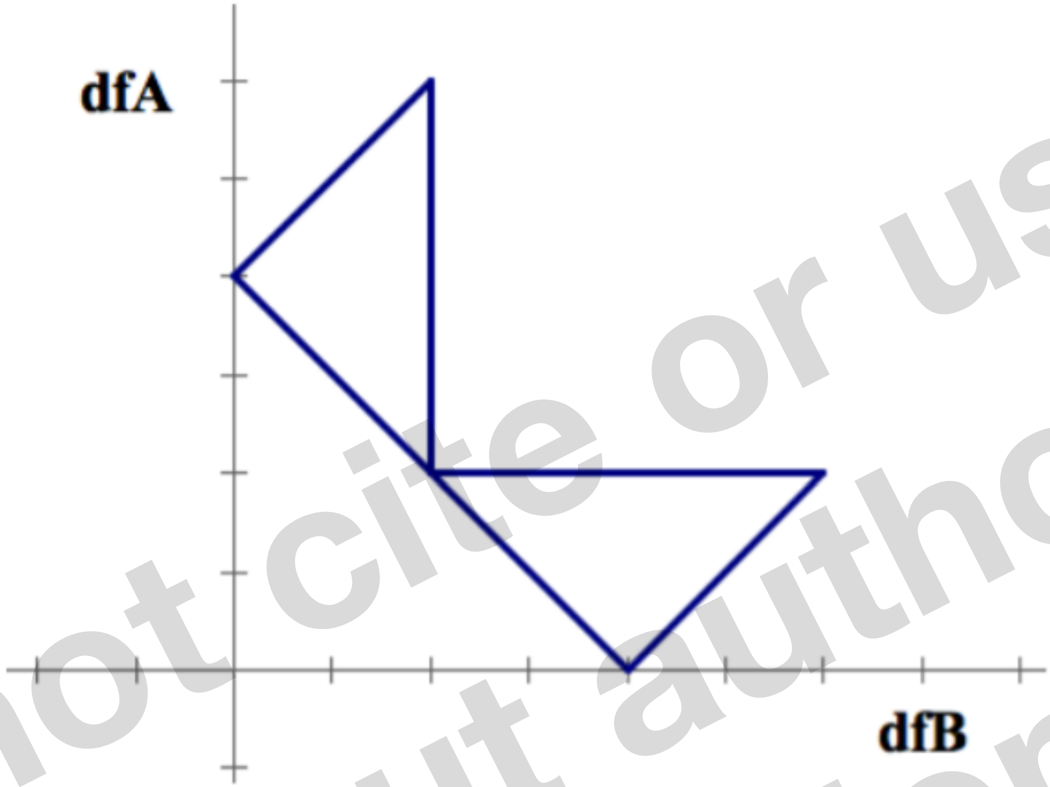
**Problem:**
A student came to you with this grap... ...ming it w...
What would you say to th... ...could that be true?





Distance from Atlanta
Distance from Gainesville
Not to Scale
Distance from Atlanta
Distance from Gainesville
Not to Scale
Distance from Atlanta
Distance from Gainesville


Drive
Reset
G_Horizontal
G_Vertical
A_Horizontal
A_Vertical
Join
Hide Link
Distance from Atlanta
Not to Scale
Distance from Gainesville
Part I-G
Part II-G
Part III-G
Part I-T
Part II-T
Part III-T
Beta 1
Beta 2
Beta 3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# THANK YOU

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The entire AR team: Irma Stevens, Biyao Liang, Teo Paoletti, Natalie Hobson, Hamilton Hardison, Katie Mauldin, Halil Tasova, and Kevin LaForest

https://sites.google.com/site/advancingreasoning/

# Assessing Secondary Teachers' Mathematical Habits of Mind

Sarah Sword
Ryota Matsuura
Miriam Gates
Al Cuoco
Glenn Stevens

DR K–12 PI Meeting
June 2, 2016

# Today's agenda

1. Background on our work
2. Paper and pencil assessment
   - Review the items in small groups
   - Whole group discussion
3. Validity and reliability of the assessment
4. Further discussion and questions

# What is ASTAHM?

***Assessing Secondary Teachers' Algebraic Habits of Mind***

ASTAHM is an NSF DRK-12 collaborative project funded in 2012 aimed at developing instruments to assess secondary teachers' mathematical habits of mind (MHoM).

## What do we mean by MHoM?

We define mathematical habits of mind (MHoM) to be:

> *the specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians.*

# Focus on MHoM

Our current focus is on three categories of MHoM:

- Seeking mathematical structure
  - Experimenting
  - Using language, notation, and pictures to acquire clarity and understanding
- Using mathematical structure
- Using mathematical language clearly (i.e., "Describing")

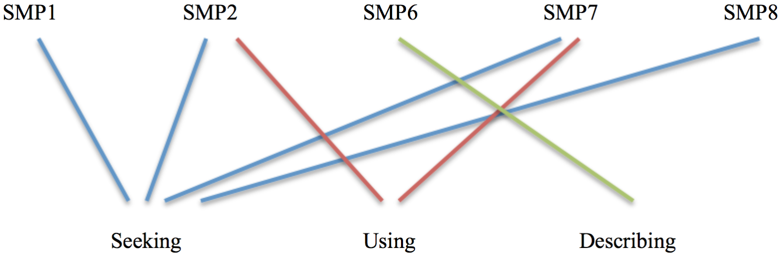
**Remark:** Focusing on three habits has allowed us to create instruments that are not too burdensome to use.

## Connection to CCSSM

Our three mathematical habits are closely related to the following Common Core Standards for Mathematical Practice:

- **MP1.** Make sense of problems & persevere in solving them
- **MP2.** Reason abstractly & quantitatively
- **MP6.** Attend to precision
- **MP7.** Look for & make use of structure
- **MP8.** Look for & express regularity in repeated reasoning

# Connection to CCSSM




We've parsed the SMPs for measurement purposes. E.g., the two processes of *seeking* and *using* structure in SMP7 look different when people do them, so we study them separately.

## MET2 framework for MKT

The *Mathematical Education of Teachers II* (MET2) framework uses four categories to characterize some of the ways in which teachers understand mathematics:

(1) As a scholar

(2) As an educator

(3) As a mathematician

(4) As a teacher

## Knowing mathematics as a mathematician

From our experience, we believe that (3) knowing mathematics as a mathematician...

- enriches and enhances the other ways of knowing mathematics,
- can bring efficiency and coherence to teachers' mathematical thinking and to their work with students,
- and thus is an important aspect of mathematical knowledge for teaching at the secondary level.

## What we're really studying

So, what we're **really** studying is the intersection of:

(3) knowing mathematics as a mathematician,

(4) knowing mathematics as a teacher.

# NCTM's Principles to Actions

## Mathematics Teaching Practices

- Establish mathematics goals to focus learning
- Implement tasks that promote reasoning & problem solving
- Use and connect mathematical representations
- Facilitate meaningful mathematical discourse
- Pose purposeful questions
- Build procedural fluency from conceptual understanding
- Support productive struggle in learning mathematics
- Elicit and use evidence of student thinking

# Initial motivation for research

- Through our PD work, we've seen that MHoM is indeed a collection of habits teachers can acquire, rather than some static you-have-it-or-you-don't way of thinking.
- Teachers report that developing these mathematical habits has a tremendous effect on their teaching.
- We recognize the need for scientific-based evidence to establish that teachers' MHoM are not static and that these habits have a positive impact on their teaching practice.
- Instruments to measure these habits have not existed.

# Research question

*What are the mathematical habits of mind that secondary teachers use, how do they use them, and how can we measure them?*

## Instruments for conducting research

To investigate our research question, we've been developing:

- Detailed definition of MHoM, based on literature, our experiences as mathematicians, and classroom observations.
- A paper and pencil (P&P) assessment that measures how teachers use MHoM when doing math for themselves.
- An observation protocol measuring the nature and degree of teachers' use of MHoM in their classroom work.

**Important:** We've seen the need for both instruments, and also the value of developing all three components together.

## What we aren't studying

There are many aspects of teaching that we value but we are *not* studying right now. For example:

- Teachers' dispositions (at least not directly)
- Teachers' beliefs
- Classroom discourse

## What we aren't creating

- Our instruments are being designed for research and development purposes, not for teacher evaluation.
- They are meant to help researchers, school leaders, professional developers, and others in better understanding and meeting the mathematical needs of secondary teachers.

# P&P assessment: Overview

- We are developing a P&P assessment that measures how teachers use MHoM when doing math for themselves.
- The assessment has been field-tested with over 500 teachers. Field-tests are ongoing.
- Initial validity and reliability testing yielded promising results. More testing is being planned.
- Again, this is a tool for research, *not* for teacher evaluation.

## P&P assessment: Key features

- Assessment measures how secondary teachers use mathematical habits of mind when doing mathematics.
- Items are accessible: most secondary teachers can solve them, or at least begin to solve them.
- Coding focuses on the *approach* to each item, not on obtaining "the correct solution."
- Assessment items are drawn from multiple sources, including our classroom observation work.

# Maximum Value

Find the maximum value of the function $f(x) = 11 - (3x - 4)^2$.

**Habit measured:** Using mathematical structure

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- Though most teachers obtained the same (correct) answer, there were vast variations in their approaches.
- These various approaches came in "clumps," as assessment experts and research literature had told us to expect.
- Using these responses, we developed a rubric that allows us to code how each teacher solved the problem.

# Sample code: SQUR

**(SQUR)** Since $(3x-4)^2$ represents the *square* of some number, it is always $\geq 0$. Thus in the function $f(x) = 11 - (3x-4)^2$, we are always subtracting a non-negative number from 11. To maximize $f(x)$, we need $(3x-4)^2 = 0$ so the max value is 11.

**Sample solution:**

$f(x) = 11 - (3x-4)^2$. Anything squared is $\geq 0$.

Therefore, $11 - (\text{stuff squared})$ must be $\leq 11$. So 11 is the max.

## Quick mathematical note

The reasoning described in SQUR depends on the fact that $x$ can be chosen so that $(3x-4)^2 = 0$. In many cases, we had no way of knowing if the teachers actually noticed this detail.

# Sample code: SYMM

**(SYMM)** Expanded $f(x)$ into $f(x) = -9x^2 + 24x - 5$. Found the axis of symmetry using the formula $x = -b/(2a) = 4/3$. Evaluated $f(4/3) = 11$ to obtain the maximum value.

**Sample solution:**

$$f(x) = 11 - (3x-4)^2$$
$$= -9x^2 + 24x - 5$$

x-coord. of vertex:

$$\frac{-b}{2a} = \frac{-24}{2(-9)} = \frac{-24}{-18} = \frac{4}{3}$$

$$f\left(\tfrac{4}{3}\right) = 11 - \left(3\left(\tfrac{4}{3}\right) - 4\right)^2$$
$$= 11 - (4-4)^2$$
$$= \boxed{11}$$

max value is 11.

## Dig into the items/rubrics

Consider these questions as you review the items/rubrics:

- Where do you see MHoM being used in these approaches?
- Do the ways in which you think about this item match the habit that we claim it measures?
- How would you want students to approach this problem?
- What connections do you see to the SMPs?
  - **MP1.** Make sense of problems & persevere in solving them
  - **MP2.** Reason abstractly & quantitatively
  - **MP6.** Attend to precision
  - **MP7.** Look for & make use of structure
  - **MP8.** Look for & express regularity in repeated reasoning

# Validity and reliability results

The current version of the assessment was administered to 274 secondary teachers. Validity and reliability tests have yielded excellent results, as summarized in the table shown.

| **Statistic** | **Value** | **Interpretation** |
|---|---|---|
| Cronbach's Alpha | 0.87 | Excellent |
| Chi-square | 29.475 ($p = 0.595$) | Good. Indicates that the model fits the data well. |
| Root mean square error of approximation (RMSEA) | 0.01 | Excellent |
| Confirmatory fit index (CFI) | 1.00 | Excellent |
| GFI (Goodness of fit index) | 0.98 | Excellent |
| Tucker-Lewis index | 1.01 | Excellent |

# Measuring teacher change[1]

Paired Samples Statistics for Teacher MHoM subscales, Time 1 and Time2 (N = 20)

| | | Mean | Std. Deviation | Std. Error Mean |
|---|---|---|---|---|
| Pair 1* | Full Assessment, Time 1 | 4.9 | 2.4 | 0.6 |
| | Full Assessment, Time 2 | 5.4 | 2.6 | 0.7 |
| Pair 2 | Using Structure, Time 1 | 4.7 | 2.8 | 0.7 |
| | Using Structure, Time 2 | 5.4 | 2.9 | 0.8 |
| Pair 3 | Language, Time 1 | 5.9 | 2.5 | 0.6 |
| | Language, Time 2 | 5.8 | 2.5 | 0.6 |
| Pair 4* | Seeking Structure, Time 1 | 4.3 | 2.6 | 0.7 |
| | Seeking Structure, Time 2 | 5.3 | 3.3 | 0.9 |

[1] Gates, M., Cordner, T., Kerins, B., Cuoco, A., Badertscher, E., & Burrill, G. (in press). Creating a hybrid immersive mathematics experience. *Mathematics Teacher*.

## Further discussion questions

- How can we ensure that we are indeed measuring MHoM and not simply capturing teachers' prior (traditional) knowledge of mathematics?
- What constitutes evidence of a "way of thinking" or "intent of an approach"? How much must a partial response include to fit into a particular category?
- What are the affordances and limitations of our instruments? What aspects of MKT are we capturing with the P&P assessment? What aspects are we missing?

## Learn more or participate

Want to use the assessment, or participate in the research? Learn more about our project at:

# mhomresearch.edc.org

If you have further feedback and/or questions, email us at:

ssword@edc.org (Sarah Sword)

# One More Item from the ASTAHM assessment: HYB[2]

To subtract a larger number from a smaller number, such as $38 - 72$, we typically "switch and negate." We first compute $72 - 38 = 34$, then negate this difference, so that $38 - 72 = -34$ (which is correct). Here is another approach, using the standard subtraction algorithm:

$$\begin{array}{r} 38 \\ -\,72 \\ \hline -46 \end{array}$$

Here, we first look at the ones place and compute $8 - 2 = 6$. Then we look at the tens place and find $3 - 7 = -4$. Lining them up, we obtain $-46$ (which is incorrect). Explain the mathematical error in this approach, i.e., why does it result in an incorrect answer?

[2]**Note:** Hy Bass suggested a version of this item.

# Our Collective Questions

- What are the relationships between MM and MHoM?
- Can connections be theoretically or practically significant?
  - What are the common/joint implications of this work?
- How do we identify (significant) MMs and MHoMs?
  - Instruction
  - Mathematics Content
  - Mathematical Practices
- How do we assess MMs and MHoMs?
- Others?