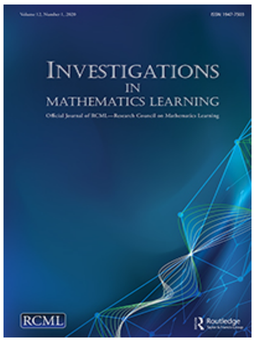


Investigations in Mathematics Learning

ISSN: 1947-7503 (Print) 2472-7466 (Online) Journal homepage: https://www.tandfonline.com/loi/uiml20

# Clarifiable Ambiguity in Classroom Mathematics Discourse

**Blake E. Peterson, Keith R. Leatham, Lindsay M. Merrill, Laura R. Van Zoest & Shari L. Stockero**

INVESTIGATIONS IN MATHEMATICS LEARNING
2020, VOL. 12, NO. 1, 28–37
https://doi.org/10.1080/19477503.2019.1619148



# Clarifiable Ambiguity in Classroom Mathematics Discourse

Blake E. Peterson[a], Keith R. Leatham[a], Lindsay M. Merrill[a], Laura R. Van Zoest[b], and Shari L. Stockero[c]

[a]Department of Mathematics Education, Brigham Young University, Provo; [b]Department of Mathematics, Western Michigan University, Kalamazoo; [c]Department of Mathematical Sciences, Michigan Technological University, Houghton, Mathematical Sciences

**ABSTRACT**

Ambiguity is a natural part of communication in a mathematics classroom. In this paper, a particular subset of ambiguity is characterized as clarifiable. Clarifiable ambiguity in classroom mathematics discourse is common, frequently goes unaddressed, and unnecessarily hinders in-the-moment communication because it likely could be made more clear in a relatively straightforward way if it were attended to. We argue for deliberate attention to clarifiable ambiguity as a critical aspect of attending to meaning and as a necessary precursor to productive use of student mathematical thinking. We illustrate clarifiable ambiguity that occurs in mathematics classrooms and consider ramifications of not addressing it. We conclude the paper with a discussion about addressing clarifiable ambiguity through seeking focused clarification.

**KEYWORDS**
Discourse; ambiguity

Researchers who look closely at the complexities of communicating in mathematics classrooms see one particular aspect of communication – sometimes referred to as *ambiguity* (e.g., Barnett-Clarke & Ramirez, 2004; Barwell, 2003; Foster, 2011) – as both inherent (and thus unavoidable) and as providing opportunities for learning. Broadly one might define ambiguity as involving "a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (Byers, 2007, p. 28). As Barwell (2003) stated, "it is the potential for ambiguity inherent in all language that allows students to investigate what it is possible to do with mathematical language, and so to explore mathematics itself" (p. 5). In fact, as Byers (2007) argued, "the power of ideas resides in their ambiguity. Thus, any project that would eliminate ambiguity from mathematics would destroy mathematics" (p. 24). Whenever students are placed in a sense-making situation, they are working with ideas they do not fully understand and, as a result, their current vocabulary is insufficient. Thus, ambiguity is a natural part of learning and an essential aspect of mathematics.

There are, however, instances of ambiguity where students *are* capable of clarifying what they *said* (although not necessarily what they *meant*). To illustrate such instances of ambiguity, consider the following example from the junior high school mathematics classroom of an award-winning teacher. While studying data about a group of bikers on a multi-day trip, students were examining a graph where distance was measured by the distance from a given city (see Figure 1). In a discussion about the graph in Figure 1, the class interpreted the plotted points at times 1.5 and 2 as an indication that the bikers were stopped on the interval between 1.5 and 2 hours. A student then volunteered, "And then they went up."

The student statement, "And then they went up," is ambiguous for a couple of reasons. First of all, it is unclear to what the student is referring by *they*. In this context *they* likely refers to either the bikers on the road or the dots on the graph, and these two interpretations have very different

**CONTACT** Blake E. Peterson peterson@mathed.byu.edu Department of Mathematics EducationBrigham Young University, 167 TMCB, Provo, UT 84602

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uiml.

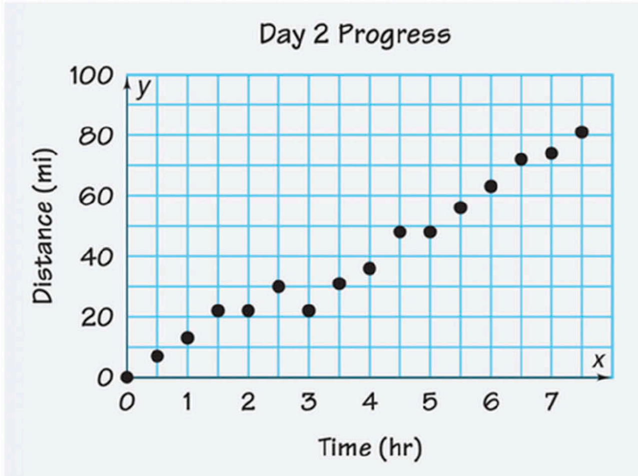


**Figure 1.** *Bikers' progress (from* Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006b, p. 12).

mathematical meanings. This ambiguity could be easily clarified were the student to indicate whether they intended the word *they* to refer to dots or bikes, and it seems likely that the student could clarify that intent. A second ambiguity in this student statement is the meaning of "went up." Of course, this meaning is likely related to the intended meaning of "they," but the elucidation of this second ambiguity is more complex and a possible site for the class to reason about the situation.

Our claim is that the first ambiguity could be clarified by the student since the student likely knows what they meant when they said "they." With such clarification, the class could then move forward in jointly making sense of the overall situation, including the current claim and the possible meanings of "went up." Without clarification of the first ambiguity, however, the communication in the classroom discourse could be hampered if some students are thinking about dots going up and other students are thinking about bikers going up. In fact, it would be difficult to make sense of "went up" without knowing whether the student was talking about dots or bikers.

In this paper, we make a theoretical argument based on our analysis of many instances of student thinking and attempt to provide insights into how one might tell the difference between the two types of ambiguity illustrated in the preceding example. Our hope in doing so is to illuminate how teachers might decide when it would be productive to push for clarification, in a sense helping to address one aspect of the teaching dilemma of deciding "whether or not to meta-comment[1]" (Chazan & Pimm, 2016, p. 28). We believe the argument we make in this paper will be useful for the mathematics education researcher studying classroom discourse, for the teacher making decisions about when to push for clarification, and for the mathematics teacher educator seeking to develop mathematics teachers' teaching practices related to productively using student mathematical thinking.

Through analyzing several thousand instances of student mathematical thinking in secondary mathematics lessons from classrooms across the US that reflected diversity of teachers, students, mathematics topics, and curricula, and hundreds of ambiguous statements from among these instances, we have come to view the type of ambiguity illustrated by the use of "they" in the phrase "And then they went up" in the following way: If a word is used such that (a) it can be interpreted in multiple viable ways, (b) the existence of those interpretations causes the overall meaning of the statement in which it occurs to be ambiguous, and (c) the individual who made the statement could

[1]In a published conversation between Pimm and Chazen (Chazan & Pimm, 2016), Pimm described meta-commenting as "a situation where some previous utterance itself (in some aspect) becomes the object of attention and conversation rather than the meaning it is intended to convey" (p. 24).

likely clarify their intended meaning for the word if asked, then the word is *clarifiably ambiguous*. Interpretations are "viable" if students in the class might reasonably infer them based on what has been discussed or is present in the current classroom dialogue. Thus, viable interpretations do not include extreme or outlying interpretations that are unlikely to exist given the context.

A teacher might infer viable interpretations through the process of *decentering* (Teuscher, Moore, & Carlson, 2016) – viewing the situation from the perspective of a typical student in the class given the context in which the part of speech is used, ranging from the context within the sentence itself to the context of what is being discussed at the moment. Viewing ambiguity from the perspective of students is critical because even if a teacher might be able to make a reasonable inference based on their experience, students in the class may not be able to make the same inference. Furthermore, although experience may aid teachers in identifying instances of clarifiable ambiguity, that same experience may lead them to make these inferences internally and thus move on without seeking explicit clarification that would allow other students in the class to have a shared sense of what has been said.

Ambiguity in classroom mathematics discourse has been a topic of a number of papers over the years (e.g., Barnett-Clarke & Ramirez, 2004; Pimm, 1987; Rathouz, 2010; Rowland, 1999). Rowland (1999, 2000) discussed how pronouns can be ambiguous, but his work focused more on the ways pronouns position people than on the ambiguous nature of the language. Although Cass (2009) discussed vague usages of various pronouns, and Barnett-Clarke and Ramirez (2004) discussed ambiguity in mathematical symbols and terms, in most cases the context around their examples seems to provide sufficient information to allow students and the teacher to accurately infer the intended referents without pushing for clarification. No literature of which we are aware explicitly addresses the distinction between ambiguity that is a natural part of the ongoing sense-making experience of classroom mathematics discourse, and clarifiable ambiguity, where clarification could be sought in order to allow the class to proceed with their mutual sense-making. We see this distinction as critical – clarification would seem to be warranted when it is unclear what a student has *said* and it is likely that, if pressed, they could simply clarify what they intended to say; after such clarification, teachers could orchestrate discussion around making sense of what the student statement might *mean*. While it may seem to be a trivial matter for experienced teachers to identify and seek clarification of clarifiable ambiguities, we identified many instances for which teachers did not seek such clarification. Regardless of experience, teachers in the classroom video that we analyzed did not identify and seek clarification of many instances that were clarifiably ambiguous.

Clarifiable ambiguity is caused, in essence, by the use of an unclear referent. This grammatical phenomenon of unclear referent is a typical topic in textbooks on grammar, but not so common in research related to oral communication and learning to write and speak in general. Research that does attend explicitly to unclear referents tends to be research about people who are acquiring a second language (Block, 1992) or who have learning challenges such as delayed development (e.g., Eigsti, de Marchena, Schuh, & Kelley, 2011) or dementia (e.g., Almor, Kempler, MacDonald, Andersen, & Tyler, 1999). The discussion of clarifiable ambiguity presented in this paper contributes to the literature related to unclear referents by discussing how one might identify and address clarifiable ambiguity during classroom mathematics discourse.

## Some Examples of Clarifiable Ambiguity

In this paper, we present theoretical ideas that both researchers and practitioners can use in their analysis of mathematics classroom discourse. Although this is a theoretical paper, it was both prompted and informed by empirical work conducted as part of the NSF-funded *Leveraging MOSTs* project (www.leveragingmosts.org). In this project, videos of the whole class discussions of 11 mathematics lessons taught by experienced teachers in various US locations (Hawaii, California, New Mexico, Utah, Mississippi, Michigan) were analyzed. A central component of project analysis involves identifying instances of student thinking worth making the object of discussion in the moment they occur during classroom discourse. As we analyze student contributions to classroom mathematics discourse, we first attempt to articulate the *student mathematics* (Leatham, Peterson, Stockero, & Van Zoest, 2015) of each instance of student thinking, or in other words, to restate the student's contribution in complete sentences or thoughts and replace pronouns and gestures with their referents when possible. In this process of articulating the student mathematics, we often find ourselves in situations where we cannot infer the student mathematics of an instance. Although sometimes this inability to infer the student mathematics stems from incomplete or inaudible statements, there are many times when student statements can be heard and seem to be complete, but wherein we still cannot make an inference. It was in this latter situation that we came to realize that our inability to infer the student mathematics is often due to what we have defined here as clarifiable ambiguity. We began to wonder about these ambiguities and their effects on classroom discourse. This paper is a result of the theoretical work that followed. The examples below are drawn from our observations in middle school mathematics lessons that were analyzed for the *Leveraging MOSTs* project. Some examples are hypothetical but based on our observations, and others come directly from actual classroom situations.

## And Then They Went Up

We begin by revisiting the introductory example "And then they went up," which was taken from an actual classroom discussion. In this case, the subject of the sentence, *they*, is unclear because of the ambiguity of its referent. Because *they* could viably be referring to the dots on the graph or to the bikers on the road, and because it is likely that the student who made the comment could succinctly clarify the interpretation, *they* is clarifiably ambiguous. If the class knew whether *they* was referring to bikers or to dots, they would be in a better position to proceed to make sense of the meaning of *went up* – the other ambiguity in this statement. If *they* refers to the dots, then the class could discuss what such a pattern in the dots might tell us about the bikers. If *they* refers to bikers, the class could discuss what students see in the data that indicates that the bikers went up. Although both conversations would focus on connections between the data in the graphical representation and the physical behavior of the bikers, knowing what *they* references would allow the teacher to know how to orchestrate that conversation – whether to help students move from dots to bikers or from bikers to dots.

## That Has a Positive Slope

In a class discussion about slopes of linear equations, a student might say, "That has a positive slope," where the clarity of this statement depends on the context in which it occurs. If there is a single linear equation on the board or being discussed, then it is likely clear what object (i.e., what equation) is being referenced and there is no ambiguity. If, however, there are linear equations on the board with both positive and negative slopes, then the student statement is ambiguous because the pronoun *that* could be referring to any of those equations. Of course, were the students to gesture to the equation they are considering, there would be no ambiguity. Without the gesture, however, there are multiple viable interpretations of the statement and the mathematical meaning behind the student statement is very different depending on which equation they are referencing. That is, the student might be providing evidence of understanding or of not understanding how to identify the slope of a line from its equation depending on which line they were referencing. The student statement is clarifiably ambiguous because it seems likely that the student could clarify which equation they are referencing.

In the example just discussed, the word *that* was intended to refer to a particular equation on the board. Suppose instead a student said, "That equation has a positive slope." The word *that* has now switched from being a demonstrative pronoun to a demonstrative adjective because it is describing the word "equation." However, this student's statement may still be ambiguous depending on the context of the situation. Even though the word *that* is coupled with "equation," if, as before, there are multiple equations on the board or being discussed in the class, there are still multiple viable interpretations of the student's statement. Thus, until the student has clarified which equation they are referencing, neither the teacher nor the class is well-positioned to make sense of the student's claim.

The case where *that* is being used as a demonstrative adjective instead of a demonstrative pronoun is "less ambiguous" in that we at least know that the subject of the sentence is an equation, we just do not know which equation. This distinction matters because, in the demonstrative pronoun situation, one might seek broader clarification of the subject of the sentence with a question such as, "*What* has positive slope?" In the demonstrative adjective situation, the clarification question could be more pointed: "*Which* equation has positive slope?" As we will argue in greater detail later in the paper, the more a teacher hones in on the part of speech that is ambiguous, the better positioned students will be to provide clarification.

## *By Dividing*

Consider the teacher-student interchange we observed when a teacher said, "What about unit rate? Could we use unit rate to solve this proportion [$6/4 = f/10$]?" and Isabella (pseudonym) responded, "Yes, by dividing." From the context, we can infer that she is saying, "We can use unit rate to solve the proportion $6/4 = f/10$ by dividing." The latter part of the sentence, however, is ambiguous; the verb *divide* has implied objects and there is no indication of what those objects are (i.e., which quantity would be divided by which other quantity). There are several legitimate possibilities for these quantities, resulting in multiple viable interpretations for this student's statement. For example, Isabella might be focused on the left-hand side of the equation. She might be saying "divide 6 by 4" to get 1.5, or she might be saying "divide the numerator by 2 and the denominator by 2" to simplify 6/4 to 3/2. Another viable interpretation would be "divide 10 by 4," which would determine the factor that could be multiplied by 6 to find $f$. Yet another viable interpretation would be "divide both sides of the equation by 1/10," which would result in multiplying both sides by 10 and yield a solution for $f$. We have articulated at least four viable interpretations for the implied objects of the verb *divide* in this context, each of which demonstrates some mathematical understanding. Since it is likely that Isabella knows the quantities that she suggested could be divided and thus could clarify those quantities, the statement "by dividing" was clarifiably ambiguous. The teacher could have zeroed in on the part of speech that created the ambiguity by asking Isabella, "What did you divide by what?" thus acknowledging the clear part of the statement – that they used division in some way – and pushing for an articulation of the unstated objects. Instead, what actually happened is that the teacher simply repeated "dividing" and then asked "Did anyone use unit rates?" seeming to dismiss Isabella's suggestion of "by dividing." By not asking Isabella to clarify what she meant by "by dividing," neither the teacher nor the students know Isabella's intended meaning. In this case, however, the ambiguity was eventually clarified, not because of a teacher move, but because of Isabella's persistence. In response to the teacher's follow up question to see if anyone had used unit rates, Isabella raised her hand and explained that she had divided 6 by 4 to get 1.5 and then multiplied 1.5 times 10 to get 15. Thus, Isabella had indeed used division as a means of determining a unit rate that she then used to solve the problem.

## *Commonality across the Examples*

Across these various examples of clarifiable ambiguity, one particular commonality stands out. Multiple viable interpretations occur when generic or implied words are used in place of more specific words. Pronouns are a wonderful tool for streamlining communication, but when their referents are unclear from the context, ambiguity occurs. Further complicating matters, the English language allows for specific subjects and objects to be completely absent from a statement, creating an even deeper layer of inference and associated possibilities for ambiguity.

### *An Instructive Non-Example*

The following example comes from an eighth-grade algebra unit. Students were learning about function composition and were given two equations: $P = 2.50V - 500$ and $V = 600 - 500R$, where profit ($P$) is related to the number of visitors ($V$) to an amusement park, and the number of visitors ($V$) is related to the probability of rain ($R$). Students were first asked to determine the profit when the probability of rain is 25% and then to find the probability of rain when the expected profit is $625 (from Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006a, p. 25). Students were able to solve for $P$ given $R$ relatively easily, but many struggled when asked to solve for $R$ given $P$. During a conversation about that struggle, a student said, "[In the first case you] just do the equation instead of doing multiple step equations." Here the student used the verb *do* in a general, colloquial way; it is not clear what she meant by "do the equation" or "doing multiple step equations," and whether each use of *do* is the same. She may have meant solve the equation(s), evaluate the equation(s), substitute something within the equation(s), or some other type of correct or incorrect equation manipulation. Because there are multiple viable options for what was meant by *do*, the student statement is ambiguous. In this example, however, the student likely used the phrase "do the equation" because she was unsure of the correct mathematical vocabulary (if there is any) to describe the particular process of "doing the equation" she was thinking about. We contend that this student is likely using the generic verb "do" to describe a process that she cannot describe with a single verb; furthermore, she may not as yet be fully aware of just what combination of solving or simplifying or evaluating or substituting she is describing. She may be able to describe the process she intended by the use of "do," but the description would not be a straightforward clarification. Thus, this example would not qualify as a clarifiable ambiguity.

That said, this student seemed to have something valuable to contribute to the mathematical conversation about students' struggles with this task, and the teacher seemed to acknowledge this by responding with a slow "okay" encouraging this student or other students to continue to share their thoughts about their struggles. However, the productivity of the subsequent discussion was hampered by the fact that the meaning of "do the equation" was and remained unclear. Since this ambiguity shares some characteristics with clarifiable ambiguity, the teacher could have better supported the classroom dialogue by approaching this ambiguity in a similar way as she would for a clarifiable ambiguity. She could have asked the student to clarify what she meant by *do* when she said, "*do* the equation." In this case, however, instead of expecting the student to easily clarify what she said, the teacher might expect the student's elaboration on her thinking to be a starting point to a class conversation. This conversation could be an opportunity to help the class better understand the differences between "solving" and "evaluating" an equation and the role "substituting into an equation" plays in either of those processes.

## Ramifications of Not Addressing Clarifiable Ambiguity

We now revisit some previously shared examples and discuss potential ramifications of leaving clarifiable ambiguity unaddressed. Consider the student claim that the proportion problem $6/4 = f/10$ can be solved "by dividing," and suppose a teacher merely accepts this response and moves on to elicit other potential strategies for solving the proportion. Now consider where various students might be with respect to making sense of the "by dividing" statement. On the one hand, some students may make an inference about which quantity would be divided by which other quantity. In making this inference, these students believe they have accurately inferred what the student meant. When the teacher moves on without addressing the ambiguity these students are likely to believe that their inference (whether accurate or not) is mathematically correct (whether it is or is not). Thus, these students might think they have a mathematical understanding of the stated strategy when they actually do not. On the other hand, other students in the class may *not* infer which quantity would be divided by which other quantity in the student's strategy. In this case, when the teacher moves on without addressing the ambiguity, the teacher communicates to this second group of students that "by dividing" was sufficiently clear. These students might believe that their lack of inference means they do not have a mathematical understanding of the stated strategy (whether they do or do not).

Whether students think they have a correct mathematical understanding when they might not, or think they do not have a correct mathematical understanding when they actually might, not seeking clarification for this ambiguous statement potentially causes an unnecessary breakdown in the classroom communication and thus impedes students' ability to make sense of the mathematics at hand. Furthermore, when a clarfiably ambiguous statement is not explicitly addressed, students are likely left unaware that what has been said is ambiguous. If the teacher implicitly infers the meaning of an ambiguous statement, students are likely not aware of the inference that the teacher has made, and thus have no idea that their interpretation of the statement may differ from that of the teacher or of other students in the class. This ramification may be particularly harmful to those learning mathematics in a second language, who might doubt both their mathematical knowledge and their ability to express it using their developing, spoken language.

In order to discuss another type of ramification, consider the initial example about the distance-time graph of a group of bikers and suppose the clarifiably ambiguous statement of "and then they went up" is left unaddressed and the conversation about the claim continues. Some students may interpret "they" as meaning the bikers while other students may interpret "they" to mean dots. If the ambiguous statement goes unaddressed, both of these particular groups of students would assume that their interpretation is correct and every subsequent statement made in the classroom would be seen in light of their own interpretation. A ramification of not addressing the imprecision in this case could be that two inconsistent parallel conversations ensue, resulting in the teacher and students talking past each other.

These two main ramifications of not addressing clarifiable ambiguity – student confusion and parallel conversations – have serious implications regarding the teacher's and students' experience in the classroom. First, students might disengage from the classroom discourse because of their inability to make sense of a clarifiably ambiguous statement. When such ambiguities occur and students are confused, or when their perceived mathematical understanding does not seem to align with the current classroom conversation, some proactive students might push on the issue until the ambiguity is cleared up and the confusion is resolved. Unfortunately, this reaction is likely the exception, as many students are unwilling to challenge a teacher or stall progressing discourse.

Second, there could be significant, detrimental repercussions for students' mathematical understanding. For instance, in the "they went up" example, failure to explicitly address the imprecision could cause or reinforce the common misconception that a graph is a picture of the physical situation (Bell & Janvier, 1981; Kozhevnikov, Hegarty, & Mayer, 2002).

Third, these ramifications may cause the teacher to miss opportunities to better understand a student's thinking, and thus miss opportunities to further that student's and the class's understanding of the mathematics at hand. Since we began to think about ambiguity as we observed it during the Leveraging MOSTs project, we particularly emphasize this implication. When a student's utterance is clarifiably ambiguous and the teacher does not address that ambiguity, the teacher is not able to confidently infer the *student mathematics* of that student's statement. Without an understanding of the student mathematics, they are not in a position to determine the potential in pursuing the student thinking (Leatham et al., 2015). Through seeking clarification, however, the teacher may very well be able to infer the student mathematics and thus be able to make decisions about the mathematical and pedagogical potential of the student's statement. We conclude the paper with a discussion about addressing clarifiable ambiguity through seeking focused clarification.

## Explicitly Addressing Clarifiable Ambiguity

Student mathematical thinking is at the heart of visions for productive classroom mathematics discourse (e.g., National Council of Teachers of Mathematics, 2014). To fully benefit from students making their thinking public, effective teachers recognize and then attend to roadblocks that hinder the effective communication of students' intended ideas. In order for students and teachers alike to make sense of each other's thinking, that stated thinking must be made clear. With a goal for such clarification in mind, teachers can attempt to internally infer what students say in order to recognize instances of ambiguity; then, when clarifiable ambiguity occurs, they can push for clarification to allow others in the classroom to also interpret what is being said. Although it seems unwise (and unnecessary) to ask for clarification about every student statement (cf. Chazan & Pimm, 2016), it seems equally unwise to never seek such clarification. We suggest that there is value in attending to ambiguity in general and, in particular, seeking to determine whether ambiguous statements are clarifiable. Some teachers, particularly novice teachers, may be reluctant to push for clarification from their students because they feel such requests may communicate a lack of mathematical understanding on their part (Peterson & Leatham, 2009). Members of a classroom community, however, can work to develop the norm that a push for clarification is not an indication of weak mathematical understanding, but rather an acknowledgment of the importance of clear communication and evidence of the centrality of students sharing their thinking to mathematics teaching and learning.

We have claimed that instances of clarifiable ambiguity are potentially productive times for teachers to push for clarification. But from our analysis of classroom mathematics discourse, just how to effectively elicit that clarification is not always obvious, even to experienced teachers. In the lesson from which we took the example "and then they went up," the teacher seemed to recognize a communication problem and pushed for clarification by asking, "What do you mean, 'They went up?'" to which a number of students responded by making hand gestures, raising their hands up as they move from left to right. Notice, however, that the students' responding gestures seem to indicate they thought they were being asked to clarify the meaning of *went up*, rather than the clarifiably ambiguous word *they*. Had the teacher honed in on the part of speech that was clarifiably ambiguous – by asking a more specific question such as, "When you say, 'they', what are you referring to?" – that student would have been better positioned to clarify their meaning, after which the sense-making discussion in the class about the meaning of *went up* could have proceeded accordingly. The teacher in this classroom is both skilled and experienced – a presidential award winner. This skill and experience, however, did not lead him to effectively seeking clarification of this clarifiable ambiguity. We believe that identifying and appropriately responding to instances of clarifiable ambiguity is possible for teachers with all levels of experience once they are attuned to these types of situations (Teuscher, Leatham, & Peterson, 2017).

Attending to the need for clarification by determining the specific source of the ambiguity (i.e., the particular part of speech that is ambiguous) in a student's statement enables teachers to ask focused, effective clarification questions. Once a teacher identifies the source of the clarifiable ambiguity, we theorize that effectively addressing it involves asking the student a clarification question that (a) explicitly establishes what the teacher *does* understand and (b) is specific to the clarifiably ambiguous part of speech.

For example, the previously suggested clarification question, "What would you divide by what?" makes it clear that the teacher understands that the student was suggesting the operation division and hones in on the teacher's desire to clarify the quantities the student envisions being involved in that operation. Similarly, asking "*Which* equation has positive slope?" establishes that the teacher has understood the students' claim that some equation has a positive slope – they just want to know which one.

Such specificity in clarification has at least four potential advantages. First, this specificity helps the student to know what aspect of their communication was seen as problematic, providing guidance for them as they seek to clarify their ideas. Second, this specificity scaffolds the entire class, better positioning everyone to engage with a clarified version of the original statement. Third, this type of clarification question gives the student, especially if they are a second-language learner, the opportunity to clarify their imprecise statement using many mathematical objects in addition to speech. Second-language learners in particular benefit from using mathematical objects such as gestures, graphs, symbols, and drawings to support their speech during mathematical discourse (Merrill, 2015). Asking a focused clarification question provides students the opportunity to use these important objects to clear up ambiguity in their language, and is an effective and equitable way to promote meaningful participation in the discourse. By explicitly addressing clarifiable ambiguity, teachers legitimize all students' efforts to make sense of others' ideas; they also model the importance of attending to clarity. Fourth, and perhaps most important, this specificity sends the message that the teacher has listened to the student, wants to understand them, and sees value in the class as a whole understanding what they are trying to communicate. Such messages play an important role in helping students to gain confidence in their abilities to contribute legitimate, useful mathematical thinking.

In conclusion, although we embrace the fact that ambiguity is a natural and important part of learning mathematics (Barwell, 2003; Byers, 2007), we see attending to clarifiable ambiguity as a critical and typically overlooked aspect of the study of the productive use of student mathematical thinking in classroom discourse. Future research could use this conceptualization of clarifiable ambiguity – both our proposed definition and our proposed approach for seeking clarification – as a tool to better understand barriers to effective classroom discourse. For example, future research could identify instances of clarifiable ambiguity in classroom data and then study how teachers respond to these instances of ambiguity and the results of those responses. Future research could also interview teachers to find out the extent to which they are aware of the different types of ambiguity that might arise in their classrooms and how they decide when to seek clarification from students. We believe that this tool is also readily accessible to teachers to use during their in-the-moment analysis of classroom discourse as a means to improve classroom communications, and thus better support the mathematical sense-making of all students in the class.

## Acknowledgments

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. Portions of this article were presented at the 2016 Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education. The authors thank Allyson Derocher and Sydney Kimball, Brigham Young University, for their contributions to the coding and analysis of the data.

## Disclosure statement

No potential conflict of interest was reported by the authors.

## Funding

This work was supported by the Division of Research on Learning in Formal and Informal Settings; U.S. National Science Foundation (NSF) 1220141, 1220357, and 1220148.

## References

Almor, A., Kempler, D., MacDonald, M. C., Andersen, E. S., & Tyler, L. K. (1999). Why do Alzheimer patients have difficulty with pronouns? Working memory, semantics, and reference in comprehension and production in Alzheimer's disease. *Brain and Language*, *67*, 202–227. doi:10.1006/brln.1999.2055

Barnett-Clarke, C., & Ramirez, A. (2004). Language pitfalls and pathways to mathematics. In R. N. Rubenstein & G. W. Bright (Eds.), *Perspectives on the teaching of mathematics* (pp. 56–66). Reston, VA: NCTM.

Barwell, R. (2003). Ambiguity in mathematics classroom discourse. In J. Williams (Ed.), *Proceedings of the British society for research into learning mathematics* (Vol. 23, pp. 1–6). Manchester, England: University of Manchester.

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics*, *2* (1), 34–42.

Block, E. L. (1992). See how they read: Comprehension monitoring of L1 and L2 readers. *Tesol Quarterly*, *26*, 319–343. doi:10.2307/3587008

Byers, B. (2007). *How mathematicians think: Using ambiguity, contradiction and paradox to create mathematics.* Princeton, NJ: Princeton University Press.

Cass, T. (2009). “It,” “that,” and “what”? Vagueness and the development of mathematical vocabulary. In B. A. Herbal-Eisenmann & M. Cirillo (Eds.), *Promoting purposeful discourse: Teacher research in mathematics classrooms* (pp. 147–165). Reston, VA: National Council of Teachers of Mathematics.

Chazan, D., & Pimm, D. (2016). Dilemmas and the teaching of mathematics. In Phakeng & Lerman (Eds.), *Mathematics education in a context of inequity, poverty and language diversity* (pp. 19–31). Cham, Switzerland: Springer.

Eigsti, I.-M., de Marchena, A. B., Schuh, J. M., & Kelley, E. (2011). Language acquisition in autism spectrum disorders: A developmental review. *Research in Autism Spectrum Disorders*, *5*, 681–691. doi:10.1016/j.rasd.2010.09.001

Foster, C. (2011). Productive ambiguity in the learning of mathematics. *For the Learning of Mathematics*, *31*, 3–7.

Kozhevnikov, M., Hegarty, M., & Mayer, R. E. (2002). Revising the visualizer-verbalizer dimension: Evidence for two types of visualizers. *Cognition and Instruction*, *20*, 47–77. doi:10.1207/S1532690XCI2001_3

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006a). *Connected mathematics 2, grade 7: Say it with symbols*. Upper Saddle River, NJ: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006b). *Connected mathematics 2, grade 8: Variables and patterns*. Upper Saddle River, NJ: Prentice Hall.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, *46*, 88–124.

Merrill, L. M. (2015). *English learners' participation in mathematical discourse* (Master's thesis). Retrieved from http://scholarsarchive.byu.edu/etd/5561

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Peterson, B. E., & Leatham, K. R. (2009). Learning to use students' mathematical thinking to orchestrate a class discussion. In L. Knott (Ed.), *The role of mathematics discourse in producing leaders of discourse* (pp. 99–128). Charlotte, NC: Information Age Publishing.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London, England: Routledge & Kegan Paul.

Rathouz, M. (2010). Ambiguity in units and their referents: Teaching and learning about rational number operations. *For the Learning of Mathematics*, *29*(1), 44–52.

Rowland, T. (1999). Pronouns in mathematics talk: Power, vagueness and generalisation. *For the Learning of Mathematics*, *19*(2), 19–26.

Rowland, T. (2000). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. London: Falmer Press.

Teuscher, D., Leatham, K. R., & Peterson, B. E. (2017). From a framework to a lens: Learning to notice student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 31–48). Cham, Switzerland: Springer International.

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, *19*, 433–456. doi:10.1007/s10857-015-9304-0